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ON THE CLOSURE OF CERTAIN ASSEMBLAGES OF TRIGONOMETRICAL FUNCTIONS

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Theorem: *Let*

$$\cos \lambda_n x \quad [0 \leq \lambda_1 < \lambda_2 < \ldots, \lambda_{k+1} - \lambda_k \geq A]$$

be a closed set of functions over $(0,\pi)$. *If*

$$P(x) = \sum_1^n a_k \cos \lambda_k x,$$

let

$$\sum_1^n a_k^2 \leq C \int_0^\pi [P(x)]^2 dx, \tag{1}$$

C being dependent only on the sequence $\{\lambda_k\}$. *Let*

$$|\lambda_k - \mu_k| \leq D < \sqrt{\frac{2A}{3C\pi^3} + \left(A + \frac{1}{2}\right)^2} - A - \frac{1}{2}. \quad [k = 1, 2, 3, \ldots].$$

Then the sequence $\cos \mu_n x$ *is closed over* $(0,\pi)$.

Proof. Let $\varphi(x)$ be any summable function of summable square over $(0, \pi)$. Let

$$\int_0^\pi \left[\varphi(x) - \sum_1^n a_k \cos \lambda_k x\right]^2 dx < e.$$

The a_k's may be so chosen because of the closure of $\{\cos \lambda_k x\}$. Let us consider

$$\psi(x) = \sum_1^n a_k (\cos \lambda_k x - \cos \mu_k x).$$

We have

$$2\psi(x) \frac{\sin x/2}{x} = \int_0^\infty F(u) \cos ux \, du,$$

where

$$F(u) = \sum_1^n a_k F_k(u).$$

$$F_k(u) = \operatorname{sgn}\left(u-\lambda_k+\frac{1}{2}\right)-\operatorname{sgn}\left(u-\lambda_k-\frac{1}{2}\right)-\operatorname{sgn}\left(u-\mu_k+\frac{1}{2}\right)+\operatorname{sgn}\left(u-\mu_k-\frac{1}{2}\right).^{1}$$

It follows that

$$\int_0^\infty 4[\psi(x)]^2\frac{\sin^2 x/2}{x^2}dx = \frac{\pi}{2}\int_0^\infty [F(u)]^2du \leq 2\pi\left(2+\frac{1+D}{A}\right)D\sum_1^n a_k^2,$$

from which we conclude that

$$\int_0^\pi\left[\psi(x)\right]^2dx \leq \frac{\pi^3}{2}\left(2+\frac{1+D}{A}\right)D\sum_1^n a_k^2.$$

Hence

$$\int_0^\pi\left[\varphi(x)-\sum_1^n a_k\cos\mu_k x\right]^2dx \leq 3\epsilon+\frac{3\pi^3}{2}\left(2+\frac{1+D}{A}\right)D\sum_1^n a_k^2$$

$$\leq 3\epsilon+\frac{3\pi^3}{2}\left(2+\frac{1+D}{A}\right)DC\int_0^\pi\left[\sum_1^n a_k\cos\mu_k x\right]^2dx$$

$$\leq 3\epsilon+\frac{3\pi^3}{2}\left(2+\frac{1+D}{A}\right)DC\left[\int_0^\pi[\varphi(x)]^2dx+2\sqrt{\epsilon}\sqrt{\int_0^\pi[\varphi(x)]^2dx}+\epsilon\right]$$

$$\leq \eta+\vartheta\int_0^\pi[\varphi(x)]^2dx,$$

where $\vartheta < 1$, and n can be made as small as is desired by taking ϵ small enough. We can hence choose our a_k's in such a way that

$$\int_0^\pi\left[\varphi(x)-\sum_1^n a_k\cos\mu_k x\right]^2dx \leq \vartheta_1\int_0^\pi[\varphi(x)]^2dx,$$

where $\vartheta_1 < 1$ and is independent of φ.

We may now choose a set of numbers b_k in such a way that

$$\int_0^\pi\left[\varphi(x)-\sum_1^n a_k\cos\mu_k x-\sum^{n_1} b_k\cos\mu_k x\right]^2dx \leq \vartheta_1\int_0^\pi\left[\varphi(x)-\sum_1^n a_k\cos\mu_k x\right]^2dx \leq \vartheta_1^2\int_0^\pi[\varphi(x)]^2dx.$$

By repeating this process, we obtain a trigonometric polynomial $P_n(x)$ in the functions $\cos\mu_k x$ such that

$$\int_0^\pi [\varphi(x)-P_n(x)]^2\,dx \leq \vartheta_1^n \int_0^\pi [\varphi(x)]^2 dx.$$

This proves the closure of the set $\cos \mu_k x$.

As a special case, let $\lambda_k = k-1$, $C = 2/\pi$. Then D can be any number less than $\sqrt{\frac{1}{3\pi^2}+\frac{9}{4}}-\frac{3}{2}$. In particular, *if*

$$\nu < \sqrt{\frac{1}{3\pi^2}+\frac{9}{4}}-\frac{3}{2}$$

the sequence

$$\cos \nu x, \cos (1+\nu)x, \cos (2+\nu)x, \ldots .$$

is closed over $(0, \pi)$.

[1] The terms at the beginning may have to be modified slightly because $F_k(u)$ is to vanish for negative u. This is allowed for in the following estimate.

ON A TYPE OF LORENTZ TRANSFORMATIONS

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1. Under a general Lorentz transformation we understand a linear transformation on the variables x, y, z, t which leaves the form

$$x^2 + y^2 + z^2 - t^2 \qquad (1)$$

invariant. Usually one type of special Lorentz transformations is considered corresponding to the case in which only t and *one* of the other variables is affected. The importance of such transformations in the Theory of Relativity is very well known. In what follows another special type of Lorentz transformations is considered, which so far as I know has not been studied, and which seems to have important applications in the study of radiation.

2. We arrive at the desired transformations in a simple way if we look for linear transformations which leave (1) invariant and, at the same time, do not change the vector 1, 0, 0, 1 and the plane $x = t$, $y = 0$. The last two conditions give four relations on the 16 coefficients of the general linear homogeneous transformation in addition to the 10 relations which express the condition that (1) remains invariant. The transformation

which we want to write down depends, therefore, on two parameters, and if we take for these two parameters the coefficients of y and z in the expression for x' we find easily that, after adjusting the directions on the y and z axes, the transformation assumes the form

$$\begin{aligned} x' &= (1-{}^1/_2(a^2+b^2))x + ay + bz + {}^1/_2(a^2+b^2)t \\ y' &= -ax + y + at \\ z' &= -bx + z + bt \\ t' &= -{}^1/_2(a^2+b^2)x + ay + bz + (1 + {}^1/_2(a^2+b^2))t \end{aligned} \tag{2}$$

3. From these formulas we can deduce formulas for point transformations of the x, y, z, t space which do not change the form

$$(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 - (t_1-t_2)^2, \tag{3}$$

the plane $x = t, y = 0$, and the vector $1, 0, 0, 1$. They are

$$\begin{aligned} x' &= (1-{}^1/_2(a^2+b^2))x + ay + bz + {}^1/_2(a^2+b^2)t + c \\ y' &= -ax + y + at \\ z' &= -bx + z + bt \\ t' &= -{}^1/_2(a^2+b^2)x + ay + bz + (1 + {}^1/_2(a^2+b^2))t + c. \end{aligned} \tag{4}$$

If we ask when a point x, y, z, t can be transformed into a point x', y', z', t' by a transformation (4) we find that a necessary condition is

$$x'-t' = x-t; \tag{5}$$

and this condition is also sufficient if $x \neq t$. If we ask what happens in case $x = t$, i.e., when a point x, y, z, x can be transformed into a point x', y', z', x' we find as a necessary and sufficient condition

$$y' = y, \quad z' = z. \tag{6}$$

The analogies with and the differences from the more familiar helicoidal motions in Euclidean space are obvious.

4. From the point of view of geometry the situation with respect to types of special Lorentz transformations may be characterized as follows. As in the case of Euclidean space (form $x^2 + y^2 + z^2 + t^2$ with four pluses) in the pseudo-Euclidean space (form (1)) a rotation always leaves two absolutely perpendicular planes in their positions (i.e., rotates them in themselves). But whereas in the Euclidean space there exists only one type of absolutely perpendicular planes—planes which intersect in a point, there are two types of absolutely perpendicular planes in pseudo-Euclidean space: outside of the "general case" in which the two planes have only one point in common there exists "a singular case" in which the two planes have a line in common, viz., a line of zero direction which is perpendicular to itself. To this singular case belongs the rotation given by the formulas (2). More general cases may be obtained by combining (2) with rotations in the x, t and the y, z planes. The two types of absolutely perpendicular

planes have been discussed by the author about two years ago;[1] it was pointed out at that time that, as a consequence, there exist two types of antisymmetric linear vector functions, and since antisymmetric linear vector functions are to be considered as infinitesimal rotations the existence of two types of rotations follows.

5. In conclusion a possible application of the transformations (4) in the theory of light might be mentioned. On the emission theory we have to consider a ray as given by a vector of square zero ("light vector") which is freely movable in the line in which it is contained, just as a particle is given by a time-like vector (four-dimensional momentum) freely movable along its line. If we want to determine the field associated with a light quantum or photon, we will want, of course, to use the same method which determines the (gravitational and electromagnetic) field associated with an electron, i.e., we will seek a field which is not changed by transformations which do not affect the light vector, and these can be deduced from (4).

[1] New York, *Trans. Amer. Math. Soc.*, **27**, January, 1925, p. 113.

CYCLICLY CONNECTED CONTINUOUS CURVES[1]

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A continuous curve M will be said to be *cyclicly connected* if and only if every two points of M lie together on some simple closed curve which is a subset of M.

The points A and B of a continuum M are said to be separated in M by the point X of M provided it is true that $M-X$ is the sum of two mutually separated sets S_1 and S_2 containing A and B, respectively. The point P of a continuous curve M is an end-point of M if and only if it is true that no simple continuous arc of M has P as one of its interior points. I have shown[2] that this definition of an end-point of a continuous curve is equivalent to the one given by Wilder.[3]

In this paper it is to be understood that the point sets considered lie in a Euclidean space of two dimensions.

THEOREM 1. *In order that a continuous curve M should be cyclicly connected it is necessary and sufficient that it should have no cut*[4] *point.*

Proof. The condition is sufficient. Let M denote a continuous curve having no cut point. Then, by a theorem due to H. M. Gehman,[5] M can have no end-point. But I have shown[2] that every continuous curve M is the sum of its cut points, end-points, and points belonging to simple

closed curves which are contained in M. It follows that every point of M belongs to some simple closed curve which is contained in M. Let A denote any definite point of M, and let N denote the set of all those points X of M such that X lies together with A on some simple closed curve in M. Clearly N exists. I shall show that N is identical with M.

I shall first show that if a simple closed curve J in M contains a point X of N then J is a subset of N. Since X belongs to N, M contains a simple closed curve C containing both A and X.

Case (1). Suppose J and C have in common at least two points. Let the two arcs of C from A to X be denoted by t_1 and t_2. On t_1 and t_2, in the order from A to X, let E and F, respectively, denote the first points belonging to J. Then if a_1 and a_2 denote the two arcs of J from E to F, every point of a_1 lies on the simple closed curve $a_1 +$ arc EAF of C, and every point of a_2 lies on the simple closed curve $a_2 +$ arc EAF of C, and both of these simple closed curves belong to M and contain A. Hence, in this case, every point of J belongs to N.

Case (2). Suppose J and C have in common only the point X. Then let M_a denote the maximal connected subset of $M-J$ which contains $C-X$. Since M has no cut point, it follows that some point Y of J, distinct from X, must be a limit point of M_a. Hence, by a theorem of R. L. Wilder's,[6] it follows that M contains an arc t from A to Y such that $t-Y$ belongs to M_a. On the arc t, in the order from A to Y, let W denote the last point belonging to C. Then if a_1 and a_2 denote the two arcs of J from X to Y, the simple closed curves $a_1 +$ arc XAW of $C +$ arc WY of t and $a_2 +$ arc XAW of $C +$ arc WY of t contain A and a_1 and A and a_2, respectively. Hence, in any case, every point of J must belong to N.

I shall next show that N is a closed set of points. Suppose, on the contrary, that there exists, in $M-N$, a point P which is a limit point of N. Let J denote a simple closed curve in M which contains P. Since P does not belong to N, it follows by what we have just shown above that no point of J can belong to N. Let M_a denote the maximal connected subset of $M-J$ which contains N.[7] Then, just as was shown above, M contains an arc t from A to some point Y of J, $Y \neq P$, such that $t-Y$ is a subset of M_a. Since M is regular and since P is a limit point of N, it follows that P can be joined in M to some point B of N by an arc t' which has no point in common with t. Let C denote a simple closed curve in M which contains both A and B. The curve C can have no point in common with J, for no point of J belongs to N. In the order from B to P on t' let E denote the last point belonging to C and let U denote the first point following E which belongs to J. In the order from A to Y on t let F denote the last point belonging to C. Let a_1 denote one of the arcs of J from U to Y. Then the simple closed curve $a_1 +$ arc YF of $t +$ arc FAE of $C +$ arc EU of t' belongs to M and contains A and a point

of J. Hence every point of J, and therefore P, belongs to N. Thus our supposition that N is not closed leads to a contradiction.

Now suppose, contrary to this theorem, that N is not identical with M. Then $M-N$ contains a point X. There exists in M an arc XA. Since N is closed, there exists on XA a point P which is, in the order from X to A, the first point belonging to N. No point of $XP-P$ belongs to N. But since P is not a cut point of M, $M-P$ contains an arc[8] t from X to A. In the order from A to X on t let E denote the last point belonging to AP, and let F denote the first point following E which belongs to PX. Then the simple closed curve formed by the interval EPF of XA plus the interval EF of t contains the point P of N and the point F which does not belong to N. But this is contrary to the fact proved above that if a single point of a simple closed curve in M belongs to N, then every point of that curve belongs to N. Thus the supposition that N is not identical with M leads to a contradiction. Hence M is a cyclicly connected continuous curve.

The condition is also necessary. For let M be a cyclicly connected continuous curve. And suppose, contrary to this theorem, that M has a cut point P. Then $M-P$ is the sum of two mutually separated point sets S_1 and S_2. Let X_1 and X_2 denote points of S_1 and S_2, respectively. The curve M contains two arcs from X_1 to X_2 which have in common only the points X_1 and X_2. But clearly this is impossible, because every arc of M from X_1 to X_2 must contain the point P. Thus the supposition that M has a cut point leads to a contradiction, and the theorem is proved.

DEFINITION. A cyclicly connected continuous curve C which is a subset of a continuous curve M will be called a *maximal cyclic curve* of M provided that C is not a proper subset of any other cyclicly connected continuous curve which is a subset of M.

THEOREM 2. *If the cyclicly connected continuous curve C is a subset of the continuous curve M, then in order that C should be a maximal cyclic curve of M it is necessary and sufficient that no connected subset of $M-C$ should have more than one limit point in C.*

Proof. The condition is sufficient. For suppose C satisfies the condition but is not a maximal cyclic curve of M. Then C must be a proper subset of some cyclicly connected continuous curve C' which is also a subset of M. Let X denote a point of C' not belonging to C, and let H_x denote the maximal connected subset of $C'-C$ which contains X. By hypothesis C can contain not more than one limit point of H_x and, since it must contain at least one, it follows that it contains exactly one limit point P of H_x. Since C' is regular, no point of H_x can be a limit point of $C'-(H_x+P)$. And no point of $C'-(H_x+P)$ is a limit point of H_x. Hence $C'-P$ is the sum of two mutually separated sets H_x and $C'-(H_x+P)$. Therefore, P is a cut point of C', contrary to theorem 1.

Thus the supposition that C is not a maximal cyclic curve of M leads to a contradiction.

The condition is also necessary. For suppose, on the contrary, that C is a maximal cyclic curve of M and that C contains two limit points A and B of some connected subset N of $M-C$. Let H denote the maximal connected subset of $M-C$ which contains N. Since M is regular, it may be shown that it contains two mutually exclusive arcs XA and YB, where X and Y are points of H. By a theorem of R. L. Moore's,[8] H contains an arc XY. The sum of the arcs XA, XY, and YB contains an arc $A'B'$, where A' and B' are points of C, such that $A'B'-(A'+B')$ is a subset of H. Since, by theorem 1, C has no cut point, the curve $C+A'B'$ can have no cut point. Hence, by theorem 1, $C+A'B'$ is a cyclicly connected continuous curve. But C is a proper subset of $C+A'B'$. Hence C is not a maximal cyclic curve of M, contrary to hypothesis. Thus the supposition that the condition of theorem 2 is not necessary leads to a contradiction, and the theorem is proved.

THEOREM 3. *If M is a continuous curve and J is any simple closed curve belonging to M, then M contains a maximal, cyclic curve which contains J.*

Proof. If M has no cut points, the truth of the theorem is an immediate consequence of theorem 1. If M has any cut points, then for each cut point X of M, let H_x denote the maximal connected subset of $M-X$ which contains $(J+X)-X$, and let N_x denote the point set H_x+X. Then for every X, N_x is a continuum containing J.

Let N denote the set of all points which are common to all of the continua N_x. The point set N is closed and contains J. Let C denote the maximal connected subset of N which contains J. The set C is a continuum. I shall show that it is a maximal cyclic curve of M.

I shall first show that if P and Q are any two points of C and t is any arc of M from P to Q, then t is a subset of C. If X is any cut point of M not on t, then H_x contains t because it contains the point P and Q. And if X is any cut point of M on t, then H_x contains $t-X$ because $t-X$ is the sum of, at most, two connected point sets each of which contains a point of H_x. Hence, in any case, t belongs to every set N_x. Therefore, it belongs to N and to C.

Since M is regular, and since C contains every arc of M joining any two points of C, it easily follows that C is regular. Hence C is a continuous curve. The curve C has no cut point. For suppose, on the contrary, that C has a cut point P. Then $C-P$ is the sum of two mutually separated point sets S_1 and S_2. Let P_1 and P_2 denote points belonging to S_1 and S_2, respectively. Then if P is a cut point of M, P_1 and P_2 lie in a connected subset H_p of $M-P$ and hence[8] can be joined by an arc t which is a subset of $M-P$. And if P is not a cut point of M, then $M-P$ is connected,

and again P_1 and P_2 can be joined by an arc t which is a subset of $M-P$. In either case, as was shown above, the arc t must belong to C. Hence S_1 and S_2 are not mutually separated, contrary to supposition. It follows that C has no cut point. Hence, by theorem 1, C is a cyclicly connected continuous curve.

The curve C is a maximal cyclic curve of M. For suppose it is not. Then by theorem 2, there exists a connected point set H lying in $M-C$ and such that C contains two limits points of H. Just as in the proof of theorem 2, it follows that M contains an arc t joining two points A and B of C and such that $t-(A+B)$ is a subset of $M-C$. But, as was shown above, C contains every arc of M which joins two points of C. Thus the supposition that C is not a maximal cyclic curve of M leads to a contradiction, and the theorem is proved.

THEOREM 4. *If the point P of a continuous curve M is neither a cut point nor an end-point of M, then M contains a maximal cyclic curve which contains P.*

Theorem 4 is a direct consequence of theorem 3 and of a theorem of mine referred to above.[2]

THEOREM 5. *No two maximal cyclic curves of a continuous curve can have more than one point in common.*

THEOREM 6. *If N denotes the point set obtained by adding together all the simple closed curves contained in a continuous curve M, then N is the sum of a countable number of maximal cyclic curves of M.*

Proof. By theorems 3 and 5, every simple closed curve J of M determines a maximal cyclic curve C_j of M which contains J. Let G denote the collection of all the maximal cyclic curves $[C_j]$ of M thus determined. I shall show that if ϵ is any positive number, then not more than a finite number of elements of G are of diameter greater than ϵ. If this is not true, then by a theorem proved by R. L. Moore and by R. G. Lubben,[9] G contains a countably infinite sequence of elements $C_1, C_2, C_3 \ldots\ldots$ all of diameter greater than some positive number ϵ and having a sequential limiting set C of diameter $\geq\epsilon$. The continuum C contains two points X and Y whose distance apart is $\geq\epsilon$. There exist two sequences of points $X_{n_1}, X_{n_2}, X_{n_3} \ldots\ldots$, and $Y_{n_1}, Y_{n_2}, Y_{n_3} \ldots\ldots$, having X and Y, respectively, as sequential limit points and such that for every positive integer i, X_{n_i} and Y_{n_i} belong to C_{n_i}. Since M is uniformly regular, it follows that for some two positive integers j and k, X_{n_j} and X_{n_k} may be joined in M by an arc t_x and Y_{n_j} and Y_{n_k} may be joined in M by an arc t_y such that t_x and t_y have no point in common. The maximal connected subset of $M-C_{n_j}$ which contains C_{n_k} minus the point, if any (see theorem 5), which C_{n_k} has in common with C_{n_j} must have at least two limit points in C_{n_j}. But this is contrary to theorem 2. Thus the supposition that

G contains infinitely many elements all of diameter greater than some positive number leads to a contradiction. Hence G is a countable collection. Clearly the sum of all the point sets of the collection G is identical with N. Thus the theorem is proved.

THEOREM 7. *Every continuous curve M is the sum of three point sets N, M_1, and M_2, where N is the sum of a countable number of cyclicly connected continuous curves no two of which have more than one point in common, M_1 is the sum of a countable number of acyclic continuous curves no two of which have more than one point in common, M_2 is a totally disconnected set of limit points of $N + M_1$, and where every point of $M_1 + M_2$ is either a cut point or an end-point of M.*

Proof. Let N denote the point set obtained by adding together all the simple closed curves contained in M. By theorem 6, N is the sum of a countable number of maximal cyclic curves of M. And by theorem 5, no two of these curves have more than one point in common.

Let us define a collection G of point sets in the following way. If E' is a maximal connected subset of $M-N$ which consists of more than one point and if E denotes the set E' plus all of its limit points, then, and only then, shall E be an element of G. Now by a theorem of mine,[10] every element E of G is an acyclic continuous curve every point of which is either a cut point or an end-point of M. From another theorem of mine[11] it readily follows that G is a countable collection. Clearly no two elements of G can have more than one point in common, because if they did their sum would contain a simple closed curve. Hence the point set obtained by adding together all the elements of G satisfies all the conditions on the set M_1 in the statement of this theorem. And if M_2 denotes the point set $M-(N + M_1)$, clearly M_2 is totally disconnected and every point of it is a limit point of $N + M_1$ and is either a cut point or an end-point of M. This completes the proof of the theorem.

THEOREM 8. *If N denotes the point set obtained by adding together all the simple closed curves contained in a continuous curve M, H_x is a maximal connected subset of N, and H denotes the set H_x plus all of its limit points, then (1) H_x is arcwise connected, (2) H is a two-way continuous curve[12] having only a countable number of cut points, and (3) the set of points $H-H_x$ is identical with the set of all the end-points of the curve H.*

Proof. (1). The set H_x is arcwise connected. For let A and B denote distinct points of H_x. Then M contains an arc t from A to B. Let K denote the set of all those points of M which separate A from B in M. Clearly K must belong to H_x and also to t. Now if P is any point of $t-K$, it is readily shown that P belongs to some simple closed curve of M and hence belongs to N. Thus every point of t belongs to N, and since H_x is the maximal connected subset of N containing A and B, it follows

that every point of t belongs to H_x. Therefore H_x is arcwise connected and contains every arc of M which joints two points of H_x.

(2). If A and B are distinct points of H and t is an arc of M from A to B, a similar argument shows that $t-(A+B)$ is a subset of N. If H_x contains no point of $t-(A+B)$ then, by a theorem of R. L. Wilder's,[3] a point of H_x can be joined to A and to B by arcs lying in M and having no point except A and B, respectively, in common with t. These arcs, together with t, form a point set containing a simple closed curve which contains t, and, therefore, A and B belong to N and hence to H_x. Therefore, by (1), t belongs to H_x. If, on the other hand, H_x contains a point of $t-(A+B)$, then it contains all of $t-(A+B)$, and, therefore, again t belongs to H. Since M is regular, it readily follows that H must also be regular. Hence H is a continuous curve. Now H can have no cut points which are not cut points of H_x. And by theorem 6, H_x is the sum of a countable number of maximal cyclic curves of M. Since, by theorem 1, no one of these curves has a cut point of itself, therefore, by a theorem of R. L. Moore's,[4] no one of them contains more than a countable number of cut points of H and, therefore, neither does their sum, H_x. Hence H has only a countable number of cut points, and, therefore,[12] it is a two-way continuous curve.

(3). Since $H-H_x$ contains neither a point of N nor a cut point of H, it follows[2] that every point of $H-H_x$ is an end-point of H. Furthermore[2] H_x contains no end-point of H. Therefore, the set $H-H_x$ is identical with the set of all the end-points of H, and the theorem is proved.

THEOREM 9. *In order that a continuous curve M should be cyclicly connected it is necessary and sufficient that the boundary with respect to*[13] *M of every connected open subset*[8] *of M which is not identical either with M or with M minus one point should consist of more than one point.*

Theorem 9 may easily be proved with the use of theorem 1.

THEOREM 10. *In order that a continuous curve M should be cyclicly connected it is necessary and sufficient that the boundary of every complementary domain of M should be a simple closed curve.*

Theorem 10 is a direct consequence of theorem 1 and of R. L. Moore's[14] theorem that in order that a continuous curve M should have no cut point it is necessary and sufficient that the boundary of every one of its complementary domains should be a simple closed curve.

[1] Presented to the American Mathematical Society, Dec. 31, 1926.

[2] Cf. my paper, "Concerning Continua in the Plane," to appear soon in *Trans. Amer. Math. Soc.*

[3] Wilder, R. L., *Fund. Math.*, **7**, 1925 (340–377).

[4] Moore, R. L., these PROCEEDINGS, **9**, 1923 (101–106).

[5] In unpublished work. This theorem is to the effect that every end-point of a continuous curve is a limit point of cut points of that curve.

[6] Wilder, R. L., *loc. cit.*, theorem 1.

[7] Clearly N is connected, because every point of N can be joined to A by a simple closed curve which is a subset of N.

[8] Cf. Moore, R. L., *Math. Zeit.*, **15**, 1922 (254–260).

[9] Cf. an abstract of a paper by the latter in the *Bull. Amer. Math. Soc.*, **32**, 1926 (14).

[10] Cf. my paper, "Concerning Continua in the Plane," *loc. cit.*, theorem 24.

[11] This theorem is to the effect that the set of all the cut points of a continuous curve cannot contain more than finite number of mutually exclusive continua of diameter greater than a given positive number, *loc. cit.*, theorem 15.

[12] A two-way continuous curve is a continuous curve having the property that every two of its points can be joined by two arcs of that curve neither of which is a subset of the other. Cf. my paper, "Two-Way Continuous Curves," *Bull. Amer. Math. Soc.*, **32**, 1926 (659–663).

[13] Moore, R. L., *Trans. Amer. Math. Soc.*, **21**, 1920 (345).

[14] Moore, R. L., *Fund. Math.*, **6**, 1925 (212).

DISPLACEMENTS IN A GEOMETRY OF PATHS WHICH CARRY PATHS INTO PATHS

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1. In a space of n dimensions, where $x^1, x^2, \ldots, x^n$ are general coordinates, the equations of the paths are

$$\frac{dx^j}{dt}\left(\frac{d^2x^i}{dt^2} + \Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) - \frac{dx^i}{dt}\left(\frac{d^2x^j}{dt^2} + \Gamma^j_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}\right) = 0, \quad (1.1)$$

where t is a general parameter and Γ^i_{jk} are functions of the x's such that $\Gamma^i_{jk} = \Gamma^i_{kj}$.

For a particular path, that is, an integral curve of equations (1.1) we have

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \varphi\frac{dx^i}{dt},$$

where φ is a determinate function of t. If we define a parameter s by

$$\frac{ds}{dt} = e^{\int \varphi dt},$$

the above equations become

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \quad (1.2)$$

Thus the parameter s, which we call the *affine parameter* of the path, is the analogue of the arc of a geodesic in a Riemannian space.

When we effect a general transformation of coördinates x^i into coördinates x'^i and require that s is not changed but that we get equations of the form (1.2) with x^i and Γ^i_{jk} replaced by x'^i and Γ'^i_{jk}, we get

$$\frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Gamma'^\alpha_{\beta\gamma} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x'^\gamma}{\partial x^k} - \Gamma^i_{jk} \frac{\partial x'^\alpha}{\partial x^i} = 0, \tag{1.3}$$

where the indices take the values 1 to n and the summation convention of a repeated index is used.[1] Functions Γ^i_{jk} and $\Gamma'^\alpha_{\beta\gamma}$ related as in (1.3) are called the coefficients of an affine connection in the two coördinate systems and the geometry based upon them, an affine geometry of paths.

2. When we do not require that s is not changed by the transformation, and consequently that equations (1.1) are transformed into analogous ones in the primes (since t is a general parameter, there is no loss of generality in using it in both sets of equations), we obtain

$$\left(\frac{\partial x'^\alpha}{\partial x^i} A^\beta_{jk} - \frac{\partial x'^\beta}{\partial x^i} A^\alpha_{jk} \right) \frac{dx^i}{dt} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0,$$

where

$$A^\alpha_{jk} = \frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Gamma'^\alpha_{\gamma\delta} \frac{\partial x'^\gamma}{\partial x^j} \frac{\partial x'^\delta}{\partial x^k} - \Gamma^i_{jk} \frac{\partial x'^\alpha}{\partial x^i}. \tag{2.1}$$

Since the above condition must be satisfied for all the paths, we must have

$$\frac{\partial x'^\alpha}{\partial x^i} A^\beta_{jk} - \frac{\partial x'^\beta}{\partial x^i} A^\alpha_{jk} + \frac{\partial x'^\alpha}{\partial x^j} A^\beta_{ki} - \frac{\partial x'^\beta}{\partial x^j} A^\alpha_{ki} + \frac{\partial x'^\alpha}{\partial x^k} A^\beta_{ij} - \frac{\partial x'^\beta}{\partial x^k} A^\alpha_{ij} = 0. \tag{2.2}$$

Multiplying by $\frac{\partial x^i}{\partial x'^\alpha}$ and summing for i and α, we obtain in consequence of (2.1)

$$(n+1) A^\beta_{jk} = \frac{\partial x'^\beta}{\partial x^j} \left(\frac{\partial \log \Delta}{\partial x^k} + \Gamma'^\alpha_{\alpha\gamma} \frac{\partial x'^\gamma}{\partial x^k} - \Gamma^i_{ik} \right) + \frac{\partial x'^\beta}{\partial x^k} \left(\frac{\partial \log \Delta}{\partial x^j} + \Gamma'^\alpha_{\alpha\gamma} \frac{\partial x'^\gamma}{\partial x^j} - \Gamma^i_{ij} \right), \tag{2.3}$$

where Δ is the jacobian $\left| \frac{\partial x'^\alpha}{\partial x^i} \right|$. When the expressions (2.3) are substituted in (2.2), the latter are satisfied identically. Hence the conditions upon the Γ's and Γ''s are given by combining (2.1) and (2.3), namely,

$$\frac{\partial^2 x'^\alpha}{\partial x^j \partial x^k} + \Pi'^\alpha_{\beta\gamma} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x'^\gamma}{\partial x^k} - \Pi^i_{jk} \frac{\partial x'^\alpha}{\partial x^i} - \frac{\partial \theta}{\partial x^j} \frac{\partial x'^\alpha}{\partial x^k} - \frac{\partial \theta}{\partial x^k} \frac{\partial x'^\alpha}{\partial x^j} = 0, \quad (2.4)$$

where

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+1}\left(\delta^i_j \Gamma^h_{hk} + \delta^i_k \Gamma^h_{hj}\right) \quad (2.5)$$

and

$$\theta = \frac{1}{n+1} \log \Delta . \quad (2.6)$$

If there is not a change of coördinates, but merely of the affine parameters, it follows that $\Pi^i_{jk} = \overline{\Pi}^i_{jk}$, where $\overline{\Pi}^i_{jk}$ is formed as in (2.5) from a set of $\overline{\Gamma}$'s. These functions Π^i_{jk} were first obtained by T. Y. Thomas in another manner[2] and were called the *coefficients of projective connections* and the geometry of such connections projective geometry of paths.

From (1.3) it follows that

$$\Gamma^i_{ij} = \Gamma'^\alpha_{\alpha\beta} \frac{\partial x'^\beta}{\partial x^j} + (n+1) \frac{\partial \theta}{\partial x^j} . \quad (2.7)$$

Moreover, when (2.7) were satisfied, equations (2.4) reduce to (1.3).

3. The equations of the preceding section may be used to define transformations of an affinely connected manifold into itself in such a manner that paths are transformed into paths. In this case the conditions of the problem are given by (2.4) on the assumption that Γ'^i_{jk} are the same functions of the x''s as the Γ's with the same indices are of the x's. If the transformations form a continuous group of r parameters, they may be considered as generated by r infinitesimal transformations. Accordingly we consider the question of infinitesimal transformations.

Consider an infinitesimal transformation defined by

$$x'^i = x^i + \xi^i \delta u, \quad (3.1)$$

where the ξ's are functions of the x's and δu is an infinitesimal. Since by hypothesis the Γ's are the same functions of the x's as the corresponding Γ''s are of the x's, the same is true of the Π's and Π''s.

Accordingly we have

$$\Pi'^i_{jk} = \Pi^i_{jk} + \frac{\partial \Pi^i_{jk}}{\partial x^h} \xi^h \delta u, \quad (3.2)$$

neglecting infinitesimals of the second and higher orders; this will be done in what follows. From (3.1) it follows that the determinant Δ of the transformation is given by

$$\Delta = 1 + \frac{\partial \xi^h}{\partial x^h}\,\delta u$$

and consequently from (2.6)

$$\frac{\partial \theta}{\partial x^i} = \frac{1}{n+1}\,\frac{\partial^2 \xi^h}{\partial x^h \partial x^i}\,\delta u\,. \tag{3.3}$$

When these values are substituted in (2.4), we obtain, on neglecting the multiplier δu,

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + \Pi^\alpha_{ik}\frac{\partial \xi^k}{\partial x^j} + \Pi^\alpha_{jk}\frac{\partial \xi^k}{\partial x^i} + \xi^h\frac{\partial \Pi^\alpha_{ij}}{\partial x^h} - \Pi^k_{ij}\frac{\partial \xi^\alpha}{\partial x^k} - \frac{1}{n+1}\left(\delta^\alpha_j\frac{\partial^2 \xi^h}{\partial x^h \partial x^i} + \delta^\alpha_i\frac{\partial^2 \xi^h}{\partial x^h \partial x^j}\right) = 0.$$

Because of (2.5) these equations become

$$\frac{\partial^2 \xi^\alpha}{\partial x^i \partial x^j} + \Gamma^\alpha_{ik}\frac{\partial \xi^k}{\partial x^j} + \Gamma^\alpha_{jk}\frac{\partial \xi^k}{\partial x^i} + \xi^h\frac{\partial \Gamma^\alpha_{ij}}{\partial x^h} - \Gamma^k_{ij}\frac{\partial \xi^\alpha}{\partial x^k} = \delta^\alpha_j\varphi_i + \delta^\alpha_i\varphi_j\,, \tag{3.4}$$

where

$$\varphi_i = \frac{1}{n+1}\left(\frac{\partial^2 \xi^h}{\partial x^h \partial x^i} + \Gamma^h_{hk}\frac{\partial \xi^k}{\partial x^i} + \xi^h\frac{\partial \Gamma^k_{ki}}{\partial x^h}\right). \tag{3.5}$$

We remark that when the first term of (3.4) is contracted for α and j, we get $(n+1)\varphi_i$. Equations (3.4) may be written in the form

$$\xi^h_{,ij} = \xi^k B^h_{ijk} + \delta^h_j\varphi_i + \delta^h_i\varphi_j\,, \tag{3.6}$$

where one or more indices preceded by a comma indicate covariant differentiation with respect to the Γ's and B^h_{ijk} are the components of the affine curvature tensor. Contracting for h and i, we have

$$\xi^i_{,ij} - \xi^h B^i_{ijh} = (n+1)\varphi_j\,. \tag{3.7}$$

If the affine parameter s is unaltered by the infinitesimal transformation, it follows from (2.7) and (3.3) that φ_i as defined by (3.5) are zero. In this case equations (3.6) become

$$\xi^h_{,ij} = \xi^k B^h_{ijk}\,. \tag{3.8}$$

According as ξ^i's satisfy (3.6) or (3.8) we say that equations (3.1) define a *projective or affine infinitesimal displacement* of the space into itself.

4. Suppose that we have a solution ξ^i of (3.6) and (3.7), and that the coördinates x^i are chosen so that in this coördinate system[3]

$$\xi^1 = 1, \quad \xi^\alpha = 0 \qquad (\alpha = 2, \ldots, n). \quad (4.1)$$

In this case equations (3.6) reduce to

$$\frac{\partial \Gamma^i_{jk}}{\partial x^1} = \delta^i_j \varphi_k + \delta^i_k \varphi_j \quad (4.2)$$

from which we have

$$\frac{\partial \Pi^i_{jk}}{\partial x^1} = 0. \quad (4.3)$$

In consequence of these equations we have:

When an affinely connected space admits a projective or affine infinitesimal displacement of the space into itself, the finite group G_1 generated by it transforms paths into paths.

In fact, for the chosen coördinate system the equations of the finite group are

$$x'^1 = x^1 + a, \qquad x'^\alpha = x^\alpha \qquad (\alpha = 2, \ldots, n)$$

where a is a parameter. For this transformation equations (2.4) reduce to $\Pi'^i_{jk} = \Pi^i_{jk}$. In consequence of (4.3) this condition is satisfied for a projective displacement. For an affine displacement we have from (4.2) that Γ^i_{jk} is independent of x^1 so that the theorem follows in this case also. Moreover, we have shown incidentally that:

The most general affinely connected manifold which admits a finite group G_1 of affine displacements into itself is given by taking for Γ^i_{jk} functions of $(n-1)$ of the coördinates.

5. The conditions of integrability of equations (3.6) are

$$\xi^h B^i_{jkl,h} - \xi^i_{,h} B^h_{jkl} + \xi^h_{,j} B^i_{hkl} + \xi^h_{,k} B^i_{jhl} + \xi^h_{,l} B^i_{jkh} = \delta^i_l \varphi_{j,k} - \delta^i_k \varphi_{j,l} + \delta^i_j (\varphi_{l,k} - \varphi_{k,l}).$$

By means of these equations it can be shown that if ξ^i_1 and ξ^i_2 are solutions of (3.6) and $X_1 f$ and $X_2 f$ are the corresponding generators, then the Poisson operator $(X_1 X_2 - X_2 X_1) f$ defines another projective or affine displacement, according as φ_i are different from zero, or are zero.

For a Riemannian geometry motions are affine displacements and the displacements which send geodesics into geodesics without preserving the arc are projective.[4] In these cases the equations are obtained by replacing Γ^i_{jk} by the Christoffel symbols of the second kind.

The question of the existence of solutions of (3.6) or (3.8), together with further developments of this theory, will be presented in a later paper.

[1] Eisenhart, L. P., and Veblen, O., these PROCEEDINGS, **8**, 1922, p. 21.

[2] These PROCEEDINGS, **11**, 1925, p. 200.

[3] Eisenhart, L. P., *Riemannian Geometry*, **1926**, p. 223.

[4] Eisenhart, L. P., *loc. cit.*, pp. 228, 233.

PHOTO-ELECTRIC EMISSION, THERMIONIC EMISSION AND PELTIER EFFECT

(FROM THE POINT OF VIEW OF DUAL ELECTRIC CONDUCTION)

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The following discussion was suggested by my recent examination of a paper[1] published several years ago by Davisson and Germer of the Bell Telephone Research Laboratory. These authors find their work, which seems to have been well done, pointing toward the view that the conducting electrons within a metal do not share the thermal activity corresponding to the temperature of the metal. Professor Millikan[2] has from experiments of his own drawn a somewhat similar conclusion. Davisson and Gerner say, "There is some difficulty, of course, in understanding how it is that, if the conduction electrons in a metal do not possess normal thermal energy, they nevertheless escape from the surface with the average energy 2 k T." They refer to a suggestion of Borelius for assistance.

I wish to show what can be done here by adopting for the moment the theory—not framed with reference to this case but from independent considerations—that the major part of the current within a metal is carried by means of electrons which go from atomic union to atomic union along the metal, without sharing the heat energy, while a much smaller part is carried by means of "free" electrons—that is, electrons free enough to have the heat energy of monatomic molecules. According to a paper which I printed in these PROCEEDINGS for March, 1921, the free electrons in tungsten at 0°C., carry only about 4% of the total electric current, the part they carry at higher temperature being still less. I do not place absolute confidence in this estimation, which was obtained by a process not wholly devoid of conjecture, but I submit that the general idea presented is worthy of tentative application in the case under discussion.

If I have found their point of view, Davisson and Germer undertook to measure the *work done against resisting forces* in taking an electron from the "free" state within the metal to the free state outside the metal, but I wish to raise the question whether this is what they really measured. I think they came much nearer to measuring the photo-electric work function—that is, the work done against resisting forces in taking an "associated conductive electron," a "valence electron" perhaps, from its atomic union within the metal to the free state outside the metal.

My argument for this opinion is as follows: If 95% (let us say) of the electrons entering the part of the filament under observation come into

it in the associated state and only 5% in the free state and if, this being the most plausible assumption we can make in the premises (application of the *mass* law), the same ratios hold where the current leaves the significant part of the filament, then 95% of the electrons emitted from the surface of the filament have to be freed from their atomic unions in order to maintain the thermionic emission and only 5% of the emitted electrons have been taken, initially, from the free state within the metal. The fact that 5% (we are assuming) of the emitted electrons enter the emitting part of the filament already free from atomic unions, and this fact alone, prevents the experiments of Davisson and Germer from giving merely the *photo-electric* work function. Naturally the value they find should be a little below the true value of this function as found from photo-electric emission, by use of the Einstein formula.

A more formal discussion follows:

Let ϕ_a be the photo-electric work function, the amount of work that must be done against resisting forces in taking an electron from the associated conductive state—a state of atomic union—within the metal to a place of given potential outside the metal. Let θ_a be the increment of heat energy, per electron, which accompanies and is additional to the work ϕ_a described above. Thus $\phi_a + \theta_a$ = the total amount of energy required, per electron, in photo-electric emission, and from the Einstein formula we have

$$\phi_a + \theta_a = h\nu \tag{1}$$

where h is the Planck constant and ν is the lowest frequency of irradiation that will produce photo-electric emission.

Let λ' be the amount of energy required to change an electron from the "associated" conductive state to the "free" conductive state within the metal. Let ϕ_f be the amount of work that must be done against resisting forces in taking an electron from the free conductive state within the metal to a place of the given potential outside the metal. Let θ_f be the increment of heat energy, per electron, which accompanies and is additional to the work ϕ_f described above.

Thus

$$\phi_a + \theta_a = \lambda' + \phi_f + \theta_f = h\nu. \tag{2}$$

The value of ν is known from direct observation for various metals at various temperatures.

Let us now have a certain length of pure metal filament under observation at a certain temperature for direct measurement of what is called the "cooling effect" of thermionic emission, as in the experiments of Davisson and Germer described in the *Physical Review* for October, **1922.**

Let k_a be the associated-electron conductivity, k_f the free-electron

conductivity, and k equal $k_a + k_f$, the total electric conductivity of the metal at the given temperature.

If the unit quantity, electromagnetic, of electricity, $(1 \div e)$ electrons, is emitted from the filament, the fraction $(k_a \div k)$ of these electrons enter the filament in the associated conductive state and the fraction $(k_f \div k)$ enter it in the free conductive state. Accordingly, if we denote by E the amount of energy required to emit the $(1 \div e)$ electrons from the surface of the filament under the given conditions, we have

$$E = [(\phi_a + \theta_a)(k_a \div k) + (\phi_f + \theta_f)(k_f \div k)](1 \div e), \qquad (3)$$

and the value of E is directly measured in the experiments indicated.

Substituting for $(\phi_f + \theta_f)$ in (3) the value $(\phi_a + \theta_a) - \lambda'$, found from (2), we get

$$\lambda'(k_f \div k) = (\phi_\alpha + \theta_\alpha) - Ee = h\nu - Ee, \qquad (4)$$

and, as all the quantities in the third member are known, the value of $\lambda'(k_f \div k)$ is thus determined for the metal under examination, at the given temperature.

Let us now suppose that we have under thermionic observation filaments of two different metals, α and β, the filaments being connected in series and maintained at the same temperature, and let the quantity E be thus determined for each metal. So we shall have two equations of the form (4), one for α and one for β,

$$[\lambda'(k_f \div k) = h\nu - Ee]_\alpha \qquad (4^\alpha)$$

and

$$[\lambda'(k_f \div k) = h\nu - Ee]_\beta \qquad (4^\beta)$$

It seems to be well established, through the work of Richardson and Compton,[3] Hennings,[4] and Millikan,[5] that for two metals joined and at the same temperature $(h\nu)_\alpha = (h\nu)_\beta$. Accordingly, we get by subtracting (4^α) from (4^β) and putting λ for $(\lambda' \div e)$,

$$[\lambda(k_f \div k)]_\beta - [\lambda(k_f \div k)]_\alpha = E_\alpha - E_\beta, \qquad (5)$$

where the value of the second member is known by direct observation.

But by reference to a paper[6] which I published several years ago it will be found that the first member of (5) is my expression for the Peltier absorption of heat at a junction where $(1 \div e)$ electrons pass from metal α to metal β. This is a quantity which can be measured independently, and thus we may have the means for an experimental test of the correctness of equation (5)—or, at least, means for checking the numerical accuracy of the experimental observations proposed.

Perhaps it would be better, instead of introducing here a previously found expression for the Peltier effect heat, to derive this expression from the conditions of the case under consideration. For this purpose we have

only to make use of the familiar logical device of an imagined reversible isothermal cycle, the net heat absorption of which must be zero.

Let us, then, suppose $(1 \div e)$ electrons to be emitted from metal β, returned by absorption into metal α and then passed by conduction through the junction from α to β, taking up in this last stage the Peltier heat $\Pi_{\alpha\beta}$. For the whole cycle we have, as the amount of heat absorbed,

$$E_\beta - E_\alpha + \Pi_{\alpha\beta} = 0, \tag{6}$$

or, see equation (5),

$$\Pi_{\alpha\beta} = E_\alpha - E_\beta = [\lambda(k_f \div k)]_\beta - [\lambda(k_f \div k)]_\alpha. \tag{7}$$

Of course the validity of equation (6) is quite independent of the truth or falsity of the dual theory of conduction. Of course, too, this whole discussion proves nothing except the fact that the dual theory of conduction enables one to form a very definite picture, whether right or wrong, of the operations under consideration.

[1] "Thermionic Work Function of Tungsten," *Physic. Rev.*, **20** (1922), p. 300.

[2] "Pulling Electrons Out of Metals by Intense Electric Fields," *Ibid.*, **27** (1926), pp. 51–67.

[3] *Phil. Mag.*, **24** (1912), p. 575.

[4] *Physic. Rev.*, **4** (1914), p. 228.

[5] *Ibid.*, **18** (1921), p. 236.

[6] "The Peltier Effect," *Proc. Nat. Acad. Sci.*, February, **1921** p. 63.

THE TRANSVERSE THERMO-ELECTRIC EFFECT IN METAL CRYSTALS

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Nearly seventy-five years ago Lord Kelvin[1] made the following prediction: "If a bar of crystalline substance possessing an axis of thermo-electric symmetry has its length oblique to this axis, a current of electricity sustained in it longitudinally will cause evolution of heat at one side and absorption of heat at the opposite side, all along the bar, when the whole substance is kept at one temperature."

Much later Voigt,[2] in developing his theory of thermo-electric action in crystals, finds the same effect in his equations, and says of it: "This transverse Peltier heat, peculiar to crystals, has not yet been investigated. But there is no reason to doubt its existence."

So far as I can find, the effect has not been reported since Voigt wrote. I have just found the effect in bismuth, zinc, tin, and cadmium. As

is to be expected, the effect is by far the largest in bismuth; in fact, it is so large that the experiment may be easily arranged to throw a sensitive galvanometer off the scale. The experiments were made on single crystal bars of approximately 10 cm. length and 6 mm. diameter; the same bars were previously used in measuring thermal conductivity.[3] Three bars of bismuth were used; in two of these the basal plane was inclined at about 20° to the length, and in the other it was parallel. The simplest possible experimental arrangement suffices to show the effect. Current connections were made at the two ends of the bar with clamps; there was an ammeter and reversing switch in the current circuit. The temperature difference between the two sides of the bar was measured with a copper-constantan thermo-element of two junctions, made of ribbon 0.002 cm. thick rolled from wire 0.010 cm. in diameter, so arranged that the two junctions were pressed by springs, one against each side of the bar, from which it was insulated by thin cellophane paper. The longitudinal positions of the two couples on the bar were adjustable independently of each other. The bar could be rotated about its longitudinal axis.

With currents of the order of 1 amp., thermal e. m. f.'s of the order of 15×10^{-4} volts were found, corresponding to a temperature difference between the two sides of the bar of 0.4°C.

A number of different sorts of checks were made to verify that the effect is the one expected. The effect reverses sign when the direction of the current is reversed; it is proportional to the current within the range investigated up to 1 amp.; it is a maximum at the two ends of that diameter which lies in the plane perpendicular to the basal plane and passing through the axis of the bar; it reverses sign if the bar is rotated through 180° beneath the couples and vanishes half way between the two orientations of maximum effect; and finally it is independent of the relative positions of the two junctions longitudinally on the bar, thus showing that the whole of one side of the bar is warmed and the other cooled, and incidentally eliminating the possibility that the effect may be due to defective insulation between the couples and the bar. The effect was of about equal magnitude in the two bismuth bars with basal plane 20° to the length, and zero in the bar with basal plane parallel to the length.

The effect was also detected in bars of zinc, cadmium, and tin, in which the axis was at approximately 45° to the length. In these metals the effect is very much less, and the disturbances from chance fluctuations in the air of the room were so great that not much more could be stated with certainty than the existence of the effect. It is of the order of 500 times smaller in these metals than in bismuth.

The existence of the effect being established, let us examine the significance of it. The theories of both Kelvin and Voigt not only demand the existence of the effect, but also give an explicit expression for its depen-

dence on the direction of the bar with respect to the crystal axis, in terms of the difference of Peltier heat for a current flowing parallel and perpendicular to the axis. These two theories indicate a close connection between the transverse thermoelectric effect and the precise way in which the Peltier heat at the end of a bar inclined at an arbitrary angle with the crystal axis varies with the angle. The expression which Kelvin and Voigt find for this latter is:

$$P_\theta = P_\parallel \cos^2\theta + P_\perp \sin^2\theta.$$

Now the argument by which Voigt derives this expression for P_θ I have already shown to be incorrect,[4] because it takes into account only the body e. m. f.'s, leaving out the surface e. m. f.'s, and in so doing failing to provide for any net e. m. f. in a complete thermo-electric circuit. The same considerations show also that Voigt's analysis for the transverse effect cannot be correct. Experimentally, I have previously found that the relation of Voigt, which is also that of Kelvin, for the dependence of P_θ on direction is not true.[5] Granting the correctness of my experimental result, the implication must then be that Kelvin's theory, as well as Voigt's, must be incorrect. Only since writing the previous paper have I found the argument of Kelvin, buried in his long paper on the "Dynamical Theory of Heat."[6] Kelvin's argument is so simple and apparently straightforward compared with that of Voigt that it is of interest to examine it in detail. The argument is made to depend on the following axiom: "Each of any number of coexisting systems of electric currents produces the same reversible thermal effect in any locality as if it existed alone." Kelvin applies the axiom to the case in point as follows. The crystal bar, cut with its length at an angle θ with the axis of rotational thermo-electric symmetry, and carrying a longitudinal current of density q, is imagined to be imbedded in some standard non-crystalline metal. The longitudinal current is now resolved into two components, one along the crystal axis and the other at right angles. Each of these virtual components flows across the transverse surface into the standard metal, and hence there are Peltier heats at this surface, which, because of the crystalline character of the metal, are different for the two components, so that there is a net effect. Now, by the condition of continuity, the components of these two components normal to the transverse surface of the bar cancel each other, so that in the metal outside only a longitudinal current flows. This current in the metal outside, parallel to the surface, may now be suppressed, as well as the external conductor itself, as having no physical connection with the phenomena inside, leaving a net Peltier heat at the transverse insulated surface of the bar. The argument easily gives for the Peltier heat per unit lateral surface the numerical expression:

$$q(P_\parallel - P_\perp) \sin\theta \cos\theta.$$

My experiments were not accurate enough to show whether Kelvin's precise expression for the variation of this transverse heat with direction holds exactly or not. The effect observed was, however, of the order of magnitude to be expected according to the formula, using in the calculation the probable value of $P_{\parallel} - P_{\perp}$ and the known thermal conductivity.

The argument and axiom of Kelvin certainly make a strong appeal, but that they cannot possibly be right may be shown by resolving the longitudinal current into another pair of directions. The two directions at 45° to the crystal axis are a pair of perpendicular directions for which the Peltier heats are equal, so that if Kelvin had resolved along these two directions instead of the two that he chose, his same argument would have given zero transverse Peltier heat. It appears in general, therefore, that Kelvin's "Principle of the Superposition of Thermo-Electric Action" cannot be maintained, and that *in general it is not allowed to resolve the current in a crystal into perpendicular components.*

We thus have the amusing situation that the transverse effect exists actually, although the theoretical prediction of it seems to be based on inadequate reasoning.

At first sight a surface heat, proportional to the volume density of current, produced by a current parallel to the surface, appears most paradoxical. In fact, if the current were physical entirely parallel to the surface, such an effect would be quite incomprehensible. But the explanation is apparent when one considers the electron structure of an electric current. In the stream of electrons which constitutes the current, and which on the whole is parallel to the surface, there are always some electrons moving obliquely. These oblique electrons are reflected on striking the surface, with change of direction with respect to the crystal axis. I have already shown[7] that in a metal crystal there is a reversible absorption or evolution of heat when a current changes its direction. Hence we are to expect a similar effect when the electrons are reflected at the surface. It is easy to see that in general, if the crystal axis is inclined to the surface there is, after reflection, a change of net direction with respect to the crystal axis of the average of all the reflected electrons and, therefore, a net heating effect, which constitutes the effect observed.

These considerations show that the precise way in which the transverse heat varies with the inclination of the bar to the crystal axis is determined by the way in which the current-carrying-electrons are distributed about the crystal axes. If the electrons move only along the axis and at right angles, then Kelvin's argument applies, and his result holds, but with the most important restriction that it is allowable to resolve the current into two rectangular components *only* along the crystal axis and at right angles. It is probable that there is a good deal of physical correctness in such a procedure. My measurements show that the Peltier heat at the end of a

bar at least does not depart by a large amount from the relation of Kelvin and Voigt, and entirely apart from such considerations, it has been suggested several times recently[8] that in a metal the electrons may flow along definite guided paths, which it is natural to suppose are simply connected with the crystal axes. A more accurate measurement of the variation of these effects with direction, which I hope to be able to make on bismuth in which the effects are largest, may be expected to throw important light on these questions.

[1] Lord Kelvin, *Mathematical and Physical Papers*, Vol. I, p. 267; *Trans. Roy. Soc. Edin.*, March, **1851.**

[2] W. Voigt,, *Lehrbuch der Kristallphysik, Teubner*, **1910,** p. 551.

[3] P. W. Bridgman, *Proc. Amer. Acad., Boston*, **61,** 1926 (101–134).

[4] Reference 3, p. 128.

[5] Reference 3, p. 129.

[6] I owe this reference to a forthcoming paper by Linder in the *Physical Review* on the t. e. m. f. of single zinc crystals.

[7] Reference 3, p. 126.

[8] H. Kamerlingh Onnes, *Rapport au IV^e Conseil Solvay;* H. A. Lorentz, *Ibid.*, also *J. Inst. Met., London*, **33,** 1925 (257–278); P. W. Bridgman, *Proc. Amer. Acad., Boston*, **60,** 1925 (442).

THE EFFECT OF VELOCITY DISTRIBUTION ON THE DEFLECTION OF ATOMS IN AN INHOMOGENOUS MAGNETIC FIELD

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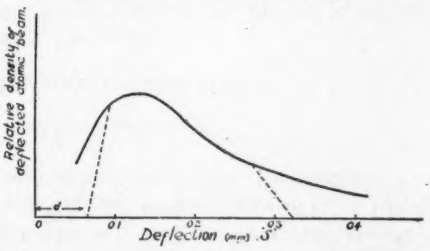
In the Stern-Gerlach experiment[1] a beam of atoms of varying velocities is deflected by an inhomogenous magnetic field. Semenoff[2] has derived an expression for the variation in the density of the image produced by the beam, with the amount of the deflection. He finds that the density of the image varies with the distance deflected according to the expression

$$e^{-\frac{\alpha}{s+d}}\left(1+\frac{\alpha}{s+d}\right)-e^{-\frac{\alpha}{s-d}}\left(1+\frac{\alpha}{s-d}\right)$$

where s is the deflection measured from the center of the undeflected beam (i.e., with the magnetic field off) d is half the width of the original beam and $\alpha=\dfrac{\mu\left(\dfrac{\partial H}{\partial s}\right)l^2}{4kT}$. In the latter expression μ is the magnetic

moment of the atom, $(\partial H/\partial s)$ the field inhomogeneity, l is the length of field traversed, k is the Boltzman constant, and T is the absolute temperature.

Semenoff raises the question as to whether the amount of deflection should be measured from the middle point of the deflected beam or from the point of maximum density. The recent determination of the magnetic moment of atomic hydrogen by Drs. Phipps and Taylor[3] in this laboratory made Semenoff's question of especial concern since it is highly desirable to estimate the possible uncertainty in their calculation. Accordingly the theoretical curve of relative density of the deflected atoms in their experiment has been worked out and plotted as a solid line in figure 1. The constants used had the following values in their apparatus: $d = 0.05$ mm.



$(\partial H/\partial s) = 1.41 \times 10^5 \frac{\text{gauss}}{\text{cm.}}$ $T = 660°$, $l = 3$ cm. The hydrogen atom was assumed to have a moment equal to one Bohr magneton.

It will be noted that the maximum of density is too flat to be located with any certainty. In the experiments of Phipps and Taylor the atomic beam was registered by a molybdenum oxide target which was reduced to an intense blue by the atomic hydrogen. The image appeared of a nearly uniform depth of color throughout, fading gradually at the edges. The location of the image as compared with the calculated deflection is shown by the dotted lines in the figure.

The fact that the image disappears rather sharply at about 0.3 mm. instead of fading gradually as the theoretical curve indicates is easily accounted for by the fact that in the calculation $\partial H/\partial s$ was assumed constant whereas it decreases rapidly with the increasing distance from the knife edge of the pole piece. The slower atoms are deflected less than would otherwise be expected.

The left-hand edge of the image does not coincide exactly with the density curve; but the uncertainties in the temperature, slit width and values of $\partial H/\partial s$ account for this discrepancy.

Thus there can be no reason to doubt that the results of Phipps and Taylor confirm the Bohr magneton value for the magnetic moment of atomic hydrogen, but it is obvious that the determination can only be made more exact by producing a beam of atoms whose velocities lie within a narrow range. Such a "monochromatic" beam could be produced by the use, in connection with a slit system, of a pair of rotating disks

with sectors cut out and synchronized in such fashion that only atoms of certain narrow ranges of velocity could pass both disks.

[1] Gerlach and Stern, *Am. Phys.*, **76**, 163 (1925); Kunz, Taylor and Rodebush, *Science*, **63**, 550 (1926); Taylor, *Physic. Rev.*, **28**, 576 (1926).

[2] Semenoff, *Zeit. Physik*, **30**, 151 (1924).

[3] *Science*, **64**, 480 (1926); *Physic. Rev.*, Feb., **1927**, to appear.

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*PINHOLE PROBE RECORD OF THE CLOSED ORGAN PIPE**

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1. *Apparatus.*—The closed organ pipe of inch brass tubing, shown in figure 1, is provided with a well-fitting, greased plug, so that the depth d may be varied at pleasure. The pipe is blown by the jet pipe j, the front of which has been compressed wedge shaped to a crack-like crevice, through which a thin sheet of air is forced, as shown by the arrow. This air lamella is cut parallel to its surface by the thin blade f, sharpened where it meets the air current. Moreover, f is adjustable on side slides (not shown), so that the width of the embouchure, e, may be varied to get the clearest tone in any case. Finally the long slender probe, bc, is inserted in the plug a and moves with it, out of the way of the air jet from j, so that the pinhole at g may always be near the bottom of the organ pipe p. The advantage of the pipe is its loud tone and the fact that air is not blown into the pipe at the embouchure. If d denotes the depth of the plug below the outlet, clear notes were obtained from the pipe as follows:

d = 15, 14.5, 13.5, 12.5, 11.5, 10.5, 9.5, 8.5, 7.5, 6.5, 5.5, 4.5, 3.5, 2.5, 1.5, 1.0, 0.5 cm.

Note: c'', $^{\sharp}c''$, d'', $^{\flat}e''$, e'', f'', g'', a'', $^{\flat}b''$, c''', d''', e''', g''', c^{iv}, e^{iv}, g^{iv}, c^{v}

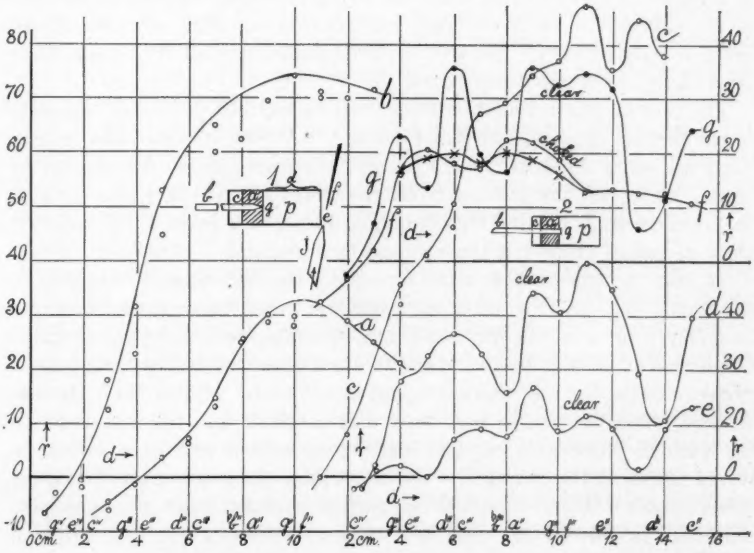
only the last being conjectural. The jet air pressure was furnished by a Fletcher blower actuated by a motor; alternatively by a rotary blower and tank with motor. In the former case the mean pressure is constant, but the running values fluctuate slightly. In the latter case pressure, though steady, is apt to decrease continually when high. The initial push of the Fletcher bellows stimulates intonations; but the acoustic pressures (which fluctuate accordingly) are largest at the final lower bellows pressure.

Acoustic pressures were measured by the interferometer U-tube. The

limits of the ocular micrometer being insufficient, the screw micrometer had to be used. These data, r, are given in 10^{-3} cm. If h is the corresponding mercury head, $\Delta h = 0.71\,\Delta r$ cm. of mercury.

The intensity of the note as registered in the graphs by the ordinate, r, depends essentially on the set of the blade, f, of the embouchure. As the best set can only be found by trial, the observations are bound to be straggling and the highest is to be taken. The graphs in many cases instance the range of fluctuation from this cause, at the same depth d of the plug. The best set was taken in each case.

On starting there is a low-pressure hiss before the pipe note breaks forth. These acoustic pressures are invariably negative and about of



the order shown at $d = 0$ (or near) of the graphs. As soon as the pipe note strikes, however, there is a definite pressure r usually in marked degree positive, as summarized by the graphs. One is inclined to regard the graphs as built up on an initially negative acoustic pressure, here about 5×10^{-3} cm. Hg.

2. *Results. Glass Pinhole Probe.*—In cases of the graphs, *a*, *b*, *c*, a sensitive glass pinhole (g) was employed. Its quill tube end required a connector tube *bc*, 6 cm. in diameter and 16 cm. long. This was joined to the U-tube by rubber tube of any convenient length. The air was fed into the jet at steady pressure.

The graphs *a*, *b*, *c*, correspond, respectively, to the lowest intermediate

and highest jet pressures available and they intersect the axis ($r = 0$) at pipe depths of 4, 2 and 1 cm. At steady low pressure the high pitch notes respond more easily, if a reflecting object like the hand is placed at the proper distance near the embouchure. Obviously the air-camella vibrates like a reed.

Each of the graphs passes through a crest or optimum about at the pipe depth $d = 10$ or 11 cm. (g'' to e''). This must be referable to the embouchure device. One notices, for instance, that at any fixed pitch (or pipe depth d) the maximum fringe displacement is gradually reached, within the lapse of seconds or even minutes. Energy is thus being continually supplied to the acoustic vibration, with the effect of steadily increasing its amptitude to a limit. This energy can only be withdrawn from the wheeze or siffle at the edge of the blade and one concludes that a mean pitch of wheeze notes is most efficiently present in correspondence with the optimum or crest of the organ pipe graph itself.

The crest in question might alternatively be ascribed to varying efficiency of the pinhole probe at different pitches. But the pinhole graphs in such cases are quite different, showing consistent sinuosity as will presently be seen. Moreover, the pinhole probe responds quite as effectively for high shrill overtones as for the strong fundamentals of the graph b, for instance. Only in case of graph c is there reason for indecision.

To test the question directly I reduced the diameter of the stem bc (originally 0.6 cm.) by inserting capillary tubes throughout its length and finally by a cotton enclosure until the time needed for a full fringe displacement (owing to the viscosity of the air conveying the acoustic pressure to the U-tube) was even a minute or more. Under these circumstances, vibration within the steam cannot occur and the treatment of the pinhole probe as a musical instrument with a pinhole embouchure would be out of the question. To my surprise these enclosures (capillary tubes of bore 0.08 and even 0.03 cm. and 13 cm. long) made no appreciable difference in the graphs obtained. There is here a mere conduction of static pressure along the stem.

3. *Pinhole in Copper Foil.*—As shown in the insert 2 at g, this was attached to the end of a quill tube, bc, 45 cm. long and 0.23 cm. in inside diameter, the pinhole moving with the plug a at the bottom of the organ pipe p.

The graphs e and d (low- and high-jet pressure) indicate that this pinhole is far less sensitive than the preceding; but both graphs are now consistently and unmistakably sinuous. The graph d probably falls increasingly to low on the left, owing to the unavoidable fall of these high-jet pressures.

In graphs d and e, an approximate location of the maxima in pipe depth would place them as follows:

maxima:	c'''	e''	$\sharp g''$	$\sharp c''$	f'''
d =	15	11	9	9	4 cm.

so that the depths are nearly as 5, 4, 3, 2, 1. The minima lie at:

minima:	$\sharp d'''$	$\flat b''$	$\sharp f''$	d''
d =	5	7.5	10	13 cm.

these being the organ-pipe notes to which the quill tube does not easily respond. They lie, in their d values, almost exactly between the maxima.

These interesting relations suggested a final experiment of direct comparison of the empty, and nearly choked quill tube (45 cm. long) as to acoustic pressure, r, generated under otherwise like conditions. The inside diameter of the quill tube, bc, figure 2, was 0.230 cm. It was closed at the pipe end with a pinhole foil g. To nearly close it, a cotton-covered straight copper wire 0.19 cm. in diameter was inserted, running from end to end. This leaves a free shell space of but 0.02 cm. thick between wire and glass, and the air resistance introduced was such that at least 1 minute was needed to obtain the maximum fringe displacement, at any pitch. It seems impossible that acoustic vibration can occur within the long rough shell in question. The Fletcher bellows was used to actuate the jet of the organ pipe, because of the fixed mean pressure to be obtained at a relatively high value.

The results are given in curve g. They are a pronounced corroboration of the curves d, e and prove that the decadence of the latter graph in high pitch is the result of falling jet pressure. There are the same crests and troughs in all curves; in curve g, however, the chief maxima are nearly of the same height.

The corresponding results, when the same quill tube probe was all but closed from end to end by the copper wire in question, are given in the graphs f, there being two different series along the main branch. This curve is quite different from curve g. The marked oscillation of the latter has largely been eliminated and f is more like the preceding graphs with glass pinholes and wide connector tubes. In other words the tendency of f is to run along the troughs of g, as if the oscillation effect of the clear quill tube were superimposed on the acoustic pressure independent of it.

4. *Conclusion.*—We may, therefore, draw the inference that the acoustic pressure at the base of the closed organ pipe may be conveyed as static pressure to the U-gauge. This pressure may be periodically (with pitch) enhanced if there is oscillation in the quill tube as well. In the case of wide connector tubes ($^1/_4$ inch) the pinhole embouchure is probably unable to stimulate appreciable vibration in the connector and the acoustic pressures obtained are without the systematic undulatory character. Hence such tubes, whether clear or partially choked from end to end, register about the same pressure. Furthermore, while the mean pressure

within the above pipe is throughout negative (below atmospheric) and constant, the relatively very large positive pressures exist within the pinhole probe and its appurtenances (quill tube and U-gauge).

* Advance note from a Report to the Carnegie Inst. of Washington, D. C.

A COMPARISON OF THE THERMIONIC AND PHOTOELECTRIC WORK FUNCTIONS FOR CLEAN TUNGSTEN

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Communicated January 2, 1927

In Richardson's thermionic equation

$$i = A\, T^{\frac{1}{2}} e^{\frac{-\phi}{kT}}$$

and in Einstein's photoelectric equation

$$\tfrac{1}{2}mv^2 = h\nu - \phi$$

ϕ is interpreted as the work necessary to carry an electron from the interior of the metal to a position outside beyond the influence of the image force. If we assume the electrons to be conduction electrons in both cases, the ϕ's should be identical, if measured at the same temperature.

The complete agreement of experiment with the thermionic equation has justified the assumption there, and Millikan[1] has shown that it is valid in photoelectric emission. However, no convincing experimental confirmation of the identity of the ϕ's has been found heretofore.

Such an experiment must be made on a metal surface, as free as possible from gas and impurities. High temperatures are necessary in order to free the metal from gas and, consequently, only tungsten, tantalum, molybdenum, and possibly platinum, are suitable. Tungsten admits of the highest temperatures and was consequently chosen for this work.

Previous Work.—Woodruff,[2] working with platinum, found that the thermionic work function was abnormally high when the specimen was insensitive photoelectrically. It is doubtful whether he had clean surfaces, however, since the thermionic work function was not the same for increasing as for decreasing temperatures, and the photoelectric sensitivity changed with time.

Harrison[3] measured the two work functions for a tungsten filament, but had the same difficulties that Woodruff found with platinum. The thermionic work function was not the same for increasing as for decreasing

temperatures. When the filament was heated and then allowed to stand at room temperature, the photoelectric sensitivity was at first quite small, but increased to large values after several days, or immediately upon the removal of liquid air from a trap. In the sensitive condition the work function was 4.57 volts (2700 Å). The thermionic work function varied between 4.8 and 6.5 volts.

Hamer[4] used a tungsten plate, which he sandpapered before mounting. He made no attempt to clean the metal by heating, and worked at pressures of the order of 1/100 mm. of mercury. He gives 4.72 volts (2615 Å) for the photoelectric work function, but did not measure the thermionic.

Hagenow[5] heated a pure tungsten plate to a bright orange-yellow and obtained a value between 5.89 (2100 Å) and 5.36 (2300 Å) for the photoelectric work function at room temperature. He found the surface only slightly sensitive, and was forced to use filters with the full light from the quartz mercury arc in making the determination. He did not measure the thermionic value.

S. C. Roy[6] has measured the variation of the total photoelectric current as a function of the temperature of the illuminating source. He finds that it obeys the equation

$$i = A\, T^{\frac{1}{2}} e^{\frac{-h\nu_0}{kT}}$$

where ν_0, the limiting frequency for the photoelectric effect, corresponds to 4.57 volts (2700 Å). He heated this specimen to a red heat and worked at quite low pressures. All the leads were brought into the tube through wax seals. The presence of any wax vapors has been found to greatly influence the thermionic work function, and consequently would be expected to effect the photoelectric. He did not measure the thermionic work function.

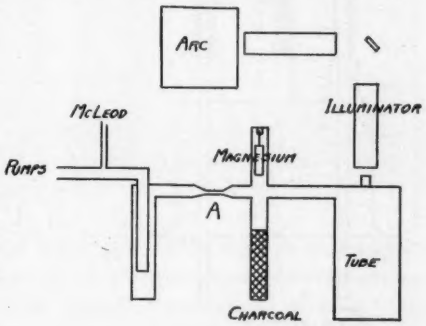


FIGURE 1

Using carefully purified tungsten, Davisson and Germer[7] have found the thermionic work function for tungsten to be 4.778 volts (2580 Å).

Dushman[8] gives the value 4.54 volts (2690 Å).

None of the photoelectric work above has been done under the extreme vacuum conditions that have been found necessary in the thermionic determinations. Langmuir[9] has shown that it is necessary to eliminate all wax vapors, bake out all glass thoroughly, and bring all metal parts to

a white heat. The author has also found that the presence of a "getter" is necessary for consistent results at temperatures below 2100°K. In the present work, consistent, reproducible thermionic results were first obtained, before any photoelectric measurements were made.

Apparatus.—All glass parts of the apparatus where of Pyrex. A graded seal from Pyrex to quartz was used to attach a quartz window. All leads through the glass were of tungsten. The arrangement of the apparatus is shown in figure 1. The arrangement of the photo-electric tube is shown in figure 2.

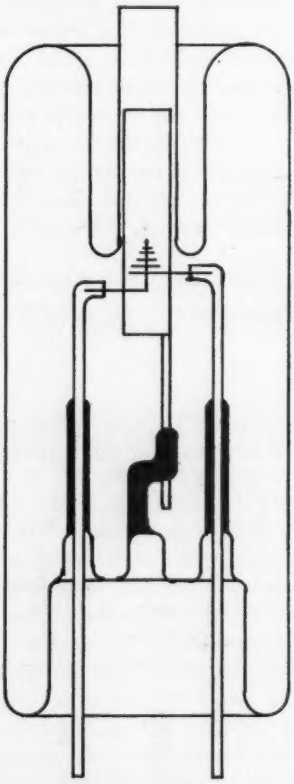


FIGURE 2

The tungsten specimen was a loose conical spiral of commercial ten mil wire, spot welded to tungsten leads. A platinum Faraday cage surrounded this, and was connected to a Dolezalek electrometer having a sensitivity of 1300 mm. per volt at 150 cm. scale distance. The electrical connections are shown in figure 3.

Illumination was from a quartz mercury

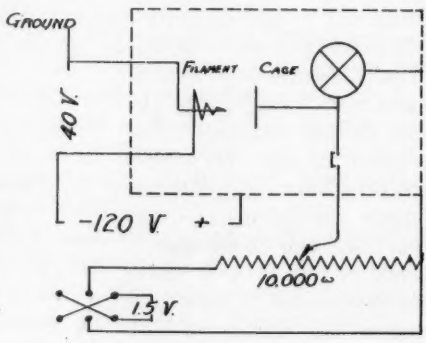


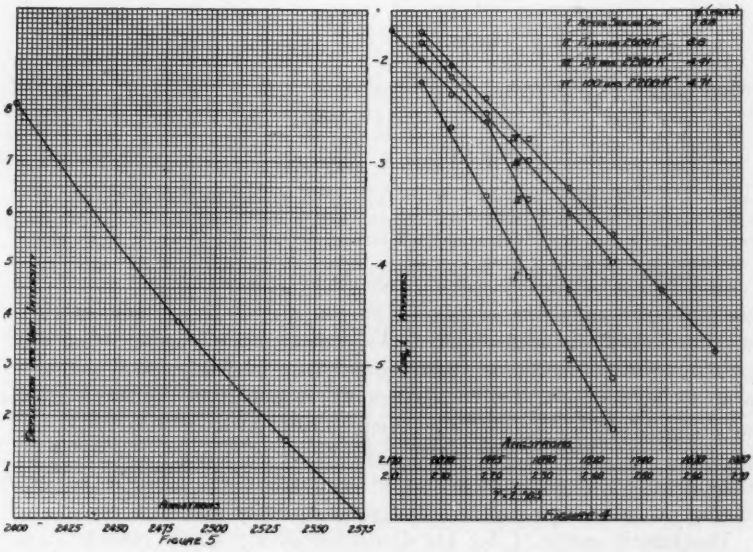
FIGURE 3

arc operating at 93 volts 2.5 amperes and 200°C. The energy distribution of such an arc has been measured by Kazda[10] in this laboratory. A Hilger mono-chromatic illuminator directed the light upon the spiral.

Results.—The tube and charcoal were baked at 500°C. for five hours. The tube was cooled and the filament glowed at 2500°K. for twenty-four hours, while the charcoal was kept at 500°C. The Faraday cage was heated to a bright yellow for ten minutes. Before installation the cage

was kept for ten minutes at a white heat in a high vacuum. The tube was again heated to 500°C. for two hours. While tube and charcoal were hot the magnesium was heated sufficiently to coat the surrounding glass. Tube and charcoal were allowed to cool, the filament and Faraday cage were reheated for a short time, and the tube sealed off the pumps at "A," figure 1. Magnesium was again vaporized and liquid air placed around the charcoal.

A filament current-temperature curve for the central turn of the spiral was made. Temperatures were measured by means of a Leeds and Northrup optical pyrometer. The transmission of the quartz window had been measured previously.



The filament was aged for 100 hours at 2200°K. with 20 milliamperes thermionic current to the Faraday cage. Measurements of the thermionic work function was made at intervals. Several of these are shown in figure 4. The values of the emission were corrected for end losses by the method of Forsythe and Worthing.[11] The final value of the work function was 4.71 volts (2620 Å).

Thermionic emission practically ceased at 1100°K., so at that temperature the photoelectric current was measured for the lines 2400, 2482 and 2536. The current per unit intensity is plotted against wave-length in figure 5. The photoelectric work function so found is 4.79 volts (2575 Å).

Conclusion.—The thermionic and photoelectric work functions for clean

tungsten have been measured, and are found to be identical within the experimental error.

[1] Millikan, R. A., *Physic. Rev.*, **18**, 1921 (236–244).
[2] Woodruff, A. E., *Physic. Rev.*, **26**, 1925 (655–670).
[3] Harrison, *Proc. Phys. Soc., London*, **38**, 1926 (214).
[4] Hamer, R., *J. O. S. and R. S. I.*, **8**, 1924 (251–257).
[5] Hagenow, C. F., *Physic. Rev.*, **13**, 1919 (415–433).
[6] Roy, S. C., *Proc. Roy Soc.*, **A112**, 1926 (599–630).
[7] Davisson, C., and Germer, L. H., *Physic. Rev.*, **20**, 1922 (300–330).
[8] Dushman, S., *Ibid.*, **25**, 1925 (338–360).
[9] Langmuir, I., *Ibid.*, **2**, 1913 (402–486).
[10] Kazda, C. B., *Ibid.*, **26**, 1925 (643–654).
[11] Forsythe, W. E., and Worthing, A. G., *Astrophys. J.*, **61**, 1925 (146–185).

THE QUANTIZATION OF THE ROTATIONAL MOTION OF THE POLYATOMIC MOLECULE BY THE NEW WAVE MECHANICS

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Communicated January 15, 1927

In a recent article[1] in these PROCEEDINGS the writer has considered the quantization of the rotational motion of the polyatomic molecule by the classical quantum theory. Since that theory is being superseded by the new wave mechanics, it seemed to be of interest to treat the same problem by the new theory. As before, the polyatomic molecule is regarded as a rigid body with three principal moments of inertia, A_x, A_y, A_z, of which A_z is assumed to be the greatest or the least. The same coordinates are used as in paper number one, and in general when the same symbol occurs in both papers it represents the same quantity.

The kinetic energy T for the dynamical system under consideration, which is easily derived,[2] is

$$T = \frac{1}{2}\left\{\frac{[p_\theta \sin\theta\cos\phi + (p_\psi - p_\phi\cos\theta)\sin\phi]^2}{A_x \sin^2\theta} + \frac{[p_\theta \sin\theta\sin\phi - (p_\psi - p_\phi\cos\theta)\cos\phi]^2}{A_y \sin^2\theta} + \frac{p_\phi^2}{A_z}\right\}. \qquad (1)$$

Here θ, ϕ, ψ are Euler's angles, and p_θ, p_ϕ, p_ψ are the conjugate momenta.

Using (1) in the Schrödinger wave equation,[3] one obtains

$$(A + b\cos 2\phi)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)$$

$$+\frac{A - b\cos 2\phi}{\sin^2\theta}\left[\frac{\partial^2 U}{\partial\psi^2} - 2\cos\theta\frac{\partial^2 U}{\partial\phi\partial\psi} + \frac{\partial^2 U}{\partial\phi^2}\right]$$

$$+ (1 + b\cos 2\phi)\frac{\partial^2 U}{\partial\phi^2} + \frac{2b\sin 2\phi}{\sin\theta}\left[\frac{\partial^2 U}{\partial\theta\partial\psi} - \cos\theta\frac{\partial^2 U}{\partial\theta\partial\phi}\right]$$

$$- 2b\cos 2\phi\cot\theta\frac{\partial U}{\partial\theta} + b\sin 2\phi(1 + 2\cot^2\theta)\frac{\partial U}{\partial\phi}$$

$$- 2b\sin 2\phi\cot\theta\csc\theta\frac{\partial U}{\partial\psi} + \lambda U = 0. \qquad (2)$$

The constants here are defined by the following equations.

$$\left.\begin{aligned} a &= \frac{1}{2}\left(\frac{1}{A_x} - \frac{1}{A_y}\right), \qquad c = \frac{1}{A_z} - \frac{1}{2}\left(\frac{1}{A_x} + \frac{1}{A_y}\right), \\ b &= \frac{a}{c}, \qquad A = \frac{1}{2c}\left(\frac{1}{A_x} + \frac{1}{A_y}\right)\lambda = \frac{8\pi^2}{h^2}\frac{E}{c} \end{aligned}\right\} \qquad (3)$$

E is the total energy of the system.

The wave equation (2), subject to the condition that U be finite and single-valued throughout all space, must be solved for the *characteristic numbers* and *characteristic functions*.

Since the mathematical work is complicated, the process of solution will merely be sketched here.

From the condition that U be single-valued in space, it follows that

$$U(\theta, \phi + 2r\pi, \psi + 2s\pi) = U(\theta, \phi, \psi) \qquad (4)$$

where r and s are any integers.

Therefore, U may be expressed as a Fourier series in ϕ and ψ. Let

$$U = \sum_{m,s=0}^{\infty} v_{m,s}\cos(m\psi + s\phi). \qquad (5)$$

The most general Fourier series would include three other types of terms beside the type given in equation (5), but actual substitution in equation (2) shows that the expression (5) serves the purpose just as well as the more general expression. The v's are functions of θ.

Substituting equation (5) in (2), equating the coefficients of the resulting Fourier series to zero, and making the change of independent variable

$$\cos\theta = 1 - 2t, \qquad (6)$$

one obtains the following doubly infinite set of ordinary differential equations.

$$bF_1\left(\frac{d}{dt}\right)v_{m,s-2} + F_2\left(\frac{d}{dt}\right)v_{m,s} + bF_3\left(\frac{d}{dt}\right)v_{m,s+2} = 0,$$

$$m, s = 0, 1, 2 \ldots . \infty \quad (7)$$

where $F_1\left(\frac{d}{dt}\right)$, $F_2\left(\frac{d}{dt}\right)$, $F_3\left(\frac{d}{dt}\right)$ are differential operators of the second order that need not be given explicitly here.

Since the only value of the subscript r of $v_{r,s}$ that occurs in (7) is $r = m$, the singly infinite set of equations obtained from (7) by assigning to m a definite value m_1 is independent of any other such set obtained by setting $m = m_2$. Therefore, the quantity m will be regarded as constant and the subscript m in $v_{m,s}$ dropped, thereby limiting the discussion to one of these singly infinite sets.

Let

$$v_{m,s} = v_s = w_s\, t^{1/2(m-s)}\,(1-t)^{1/2(m+s)}. \quad (8)$$

Then the set of equations (7) reduce to

$$\frac{b}{2}\left\{t^2\frac{d^2w_{s-2}}{dt^2} + 2(m-s+2)t\frac{dw_{s-2}}{dt} + (m-s+1)(m-s+2)w_{s-2}\right\}$$

$$+\left\{At(1-t)\frac{d^2w_s}{dt^2} + A\left[(m-s+1) - 2(m+1)t\right]\frac{dw_s}{dt}\right.$$

$$\left. + \left[\lambda - s^2 - Am(m+1)\right]w_s\right\}$$

$$+\frac{b}{2}\left\{(1-t)^2\frac{d^2w_{s+2}}{dt^2} - 2(m+s+2)(1-t)\frac{dw_{s+2}}{dt}\right.$$

$$\left. + (m+s+1)(m+s+2)w_{s+2}\right\} = 0,$$

$$s = 0, 1, 2, 3, \ldots . \infty. \quad (9)$$

The set of equations (9) will be solved by a perturbation method. Setting $b = 0$ reduces the asymmetrical case of the polyatomic molecule to the symmetrical case, which has been solved by Reiche.[4] One solution in that case is

$$\left.\begin{aligned} w_n = w_{n,0} = 1, \qquad w_s = w_{s,0} = 0 \qquad \text{when } s \neq n \\ \lambda = \lambda_0 = Am(m+1) + n^2 \end{aligned}\right\} \quad (10)$$

where m and n are positive integers, and $n \leqq m$. Due to the fact that the dynamical system is degenerate all the other solutions given by Reiche

are equivalent to this one. This particular one is chosen because of its simplicity.

Let us assume, therefore, that

$$\left.\begin{aligned} w_s &= w_{s,0} + w_{s,1}b + w_{s,2}b^2 + \ldots . \\ \lambda &= \lambda_0 + \lambda_1 b + \lambda_2 b^2 + \ldots . \end{aligned}\right\} \quad (11)$$

Substituting (11) in (9), and equating to zero the coefficients of the resulting power series in b, one obtains

$$\begin{aligned} &\frac{1}{2}\left\{ t^2 \frac{d^2 w_{s-2,i}}{dt^2} + 2(m-s+2)t\frac{dw_{s-2,i}}{dt} \right. \\ &\qquad \left. + (m-s+1)(m-s+2)w_{s-2,i} \right\} \\ &+ \left\{ At(1-t)\frac{d^2 w_{s,i+1}}{dt^2} + A\left[(m-s+1) \right.\right. \\ &\qquad \left.\left. - 2(m+1)t \right]\frac{dw_{s,i+1}}{dt} + \left[\lambda_0 - s^2 - Am(m+1) \right] w_{s,i+1} \right. \\ &\left. + \lambda_1 w_{s,i} + \lambda_2 w_{s,i-1} + \ldots . \lambda_{i+1} w_{s,0} \right\} \\ &\qquad + \frac{1}{2}\left\{ (1-t)^2 \frac{d^2 w_{s+2,i}}{dt^2} - 2(m+s+2)(1-t)\frac{dw_{s+2,i}}{dt} \right. \\ &\left. + (m+s+1)(m+s+2)w_{s+2,i} \right\} = 0 \end{aligned}$$

$$i, s = 0, 1, 2, 3, \ldots . \ \infty \quad (12)$$

and the set of equations

$$\begin{aligned} &At(1-t)\frac{d^2 w_{s,0}}{dt^2} + A\left[(m-s+1) - 2(m+1)t \right]\frac{dw_{s,0}}{dt} \\ &\qquad \left[\lambda_0 - s^2 - Am(m+1) \right] w_{s,0} = 0 \end{aligned}$$

$$s = 0, 1, 2, 3, \ldots . \ \infty . \quad (13)$$

Equations (13) will be called set zero, and in general the equations of (12) for which $i = r-1$ will be called the rth set of equations.

The solution of set zero is given in (10). Using that solution, one can obtain the solution of set 1, thence the solution of set 2, etc. Thus step by step the equations (12) may be solved. The entire solution is of the form

$$\left.\begin{aligned} w_{s,i} = \gamma_{s,i}, \text{ a constant, if } s &= n, n \pm 2, \ldots . (n \pm 2i) \\ w_{s,i} = 0 \text{ , if } s &\neq n, n \pm 2, \ldots . (n \pm 2i) \end{aligned}\right\} \quad (14)$$

$$\lambda_{2i+1} = 0, \; i = 0, 1, 2, \ldots \infty \tag{15}$$

$$\lambda_2 = \frac{1}{16}\left\{\frac{(m-n+1)(m-n+2)(m+n)(m+n-1)}{n-1} - \frac{(m+n+1)(m+n+2)(m-n)(m-n-1)}{n+1}\right\} \tag{16}$$

if $n \geq 2$. If $n < 2$, the first term in the brackets must be omitted.

$$\lambda_4 = -\frac{\lambda_2}{64}\left[\frac{(m+n)(m+n-1)(m-n+1)(m-n+2)}{(n-1)^2} + \frac{(m-n)(m-n-1)(m+n+1)(m+n+2)}{(n+1)^2}\right] + \frac{1}{2048}\left[\frac{(m+n)!\;(m-n+4)!}{(m+n-4)!\;(m-n)!\;(n-1)^2(n-2)} - \frac{(m-n)!\;(m+n+4)!}{(m-n-4)!\;(m+n)!\;(n+1)^2(n+2)}\right], \tag{17}$$

if $n \geq 4$. If $n < 4$, the term having $(n-2)$ in the denominator must be omitted, and if $n < 2$, the terms having $(n-1)$, in the denominator must also be omitted.

By the process here sketched as many of the λ_i's may be determined as is desired.

Using these results in equations (3) and (11),

$$E = \frac{h^2}{8\pi^2}\left\{\frac{1}{2}\left(\frac{1}{A_x} + \frac{1}{A_y}\right)m(m+1) + cn^2 + \sum_{r=1}^{\infty} f_r(m,n)\frac{a^{2r}}{c^{2r-1}}\right\}, \tag{18}$$

which is analogous to equation (22) of paper number one.

$$f_r(m,n) \equiv \lambda_{2r} \; . \tag{19}$$

If $n \geq 2$, λ_2 may be reduced to the form

$$\lambda_2 \equiv f_1 = \frac{1}{8}\left[\frac{m^2(m+1)^2}{n^2-1} + 2m(m+1) - 3n^2\right], \tag{20}$$

and if $n \geq 4$,

$$\lambda_4 \equiv f_2 = \frac{1}{512}\left[\frac{m^4(m+1)^4}{(n^2-1)^2}\left(\frac{9}{n^2-4} - \frac{4}{n^2-1}\right)\right.$$

$$- 12\frac{m^3(m+1)^3}{(n^2-1)(n^2-4)}$$
$$+ m^2(m+1)^2\left(\frac{18}{n^2-1}+\frac{4}{n^2-4}\right)$$
$$+ 36m(m+1) - 51n^2\Big] \qquad (21)$$

The expression for f_1 and f_2 in equations (20) and (21) should be compared with the quantities designated by the same symbol in paper number one. It will be noted that if the substitutions

$$m^2 \sim m(m+1)$$
$$n_2 \sim n^2, (n^2-1), (n^2-4), \text{ etc.}$$

be made in equations (20) and (21), f_1 and f_2 reduce to the corresponding quantities in the classical quantum theory.

In a paper to be published elsewhere a more detailed treatment of the preceding mathematical theory will be given.

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[1] Witmer, E. E., *Proc. Nat. Acad. Sci.*, **12**, 602, 1926. This paper will be referred to hereafter as paper number one.

[2] Cf. Born, *Atommechanik*, p. 30.

[3] Schrödinger, E., *Ann. Physik*, **79**, 748, 1926. Of the various formulations of the wave equation the one given in this reference is the most convenient for use in this problem.

[4] Reiche, F., *Zeit. Physik*, **39**, 444, 1926.

A NOTE ON THE SPECTRUM OF DOUBLY IONIZED SCANDIUM

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Some of the spectral terms of Sc III have recently been given by Gibbs and White.[1] In their scheme for this spectrum these authors have taken λλ 2734, 2699 as the first pair of the principal series $4^2P_{12}-4^2S_1$, the triplet at λλ 1598, 1603, 1610 as an inverted first member of the diffuse series $4^2P_{12}-3^2D_{23}$ and finally λλ 2012, 1993 as $4^2P_{12}-5^2S_1$. It seemed to the writer that this last choice was not particularly probable on account of the intensity relations of the lines, the shorter λ 1993 having a much larger intensity than that of the longer λ 2012. Some scandium plates taken by Dr. R. J. Lang with a vacuum spectrograph already described[2] were, therefore, examined in an attempt to throw some light on this dis-

crepancy. The grating used has a radius of 6 feet with 30,000 lines to the inch giving a dispersion of about 4.6 Å per mm. in the first order. The method and the standards used for computing the wave-lengths have been given in a previous paper.[3] The aluminum line 1862.78 and the oxygen line 834.46 determined by Bowen and Ingram[4] were also used as standards. The computed wave-lengths are given in the following table.

SPECTRUM OF Sc III

INT.	λ I. A. VAC	ν	dν	TERM COMBINATION	TERM VALUES
0	730.60	136873.8		$3^2D_2-4^2F_3$	
			198.3		
1	731.66	136675.5		$3^2D_3-4^2F_4$	
					$4^2S_1 = 173906.3$
5	1598.06	62575.8	197.4	$3^2D_2-4^2P_2$	
10	1603.12	62378.4		$3^2D_3-4^2P_2$	$4^2P_1 = 137340.8$
			473.7		
8	1610.25	62102.1		$3^2D_2-4^2P_1$	$4^2P_2 = 136867.1$
4	1993.96	50151.5		$4^2P_1-4^2D_2$	$3^2D_2 = 199443.0$
			473.2		
6	2011.13	49723.3	45.0	$4^2P_2-4^2D_3$	$3^2D_3 = 199245.5$
1	2012.95	49678.3		$4^2P_2-4^2D_2$	
					$4^2D_2 = 87188.8$
*10	2699.84	37039.2		$4^2S_1-4^2P_2$	$4^2D_3 = 87143.8$
			473.7		
* 8	2734.82	36565.5		$4^2S_1-4^2P_1$	Assumed value of
*2u	4062.48	24616.3		$4^2D_2-4^2F_3$	
			44.7		$4F = 62570$
*2u	4069.88	24571.6		$4^2D_3-4^2F_4$	

* Measured by Exner and Haschek but reduced to λ I.A. vac.

From the appearance of the lines on the plate and for the reasons given below it was concluded that λλ 2012, 1993, together with λ 2011 form a triplet in which λ 2012 is the satellite of the strongest member λ 2011. If the term values of KI, CaII and ScIII progress in the manner which is usual for stripped atoms, i.e., $\sqrt{\frac{\nu}{R}}$ plotted against atomic number gives, approximately, a Moseley curve but for which the slope gradually decreases with increasing atomic number, then $3D$ and $4D$ for ScIII should have values about 200000 and 90000, respectively. If $3D-4P$ is 60000, then $4P-4D = 3D-4D-(3D-4P) = 50000$. It, therefore, seems very probable that the triplet at λ 2011 is the first normal member of the diffuse series of ScIII. A search was then made for the first member of the first fundamental series $3^2D_{23}-4^2F_{34}$. Assuming for $4F$ the value 62570 which is suggested by the values of the $4F$ terms of KI and CaII, the predicted value of the wave number for $3D-4F$ is 137430. A pair was

found at λλ 730.60, 731.66 having approximately this value for ν, the separation being $d\nu = 198.3$ which is very close to the value $3^2D_2-3^2D_3$, found for the inverted triplet at λ 1600, viz., $d\nu = 197.4$. This pair is, therefore, considered to be $3^2D_{23}-4^2F_{34}$, the satellite $3^2D_3-4^2F_3$ not appearing. The intensity of the spectra formed by the grating falls off very rapidly below 800 Å; this accounts for the rather unexpectedly small intensity for this pair. Taking 62570 for the value of 4 F the term values given in the table were calculated. A very satisfactory piece of evidence in support of this interpretation of the spectrum of ScIII is the very close agreement between the predicted value $\nu = 24573.8$ for $4^2D_3-4^2F_4$ and the observed wave number 24571.6 of the longer wave-length component of a doublet having the expected separation $d\nu = 44$ measured by Exner and Haschek.[5] This pair has therefore been taken to be $4^2D_{23}-4^2F_{34}$. The satellite $4^2D_3-4^2F_3$ does not appear.

[1] Gibbs and White, *Proc. Nat. Acad. Sci.*, **12**, 598, 1926.
[2] Lang and Smith, *J. O. S. A. and R. S. I.*, **12**, 523, 1926.
[3] Smith and Lang, *Physic. Rev.*, **28**, 36, 1926.
[4] Bowen and Ingram, *Ibid.*, **28**, 444, 1926.
[5] *Handbuch der Spectroscopie*, Kayser, Vol. 6.

THE THERMAL DECOMPOSITION OF AMMONIA UPON THE SURFACE OF A MOLYBDENUM WIRE

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The thermal decomposition of ammonia has been previously investigated upon surfaces of fused quartz, platinum and tungsten.[2] The decomposition of ammonia upon molybdenum, now reported, is different from that upon any of the previous catalysts and reveals factors of interest to the general study of contact catalysis.

Experimental Method.—The catalyst consisted of a molybdenum wire 0.005 cm. in diameter mounted between platinum leads in a bulb, as shown in figure 1. The bulb was made from a large slip joint. The lower lead dipped into a pool of mercury so that the bulb could be removed without dismounting the wire or a new wire could be inserted with ease.

The potential was measured only across the hot portion of the filament by means of fine molybdenum connections. From the ratio of the potential, as thus measured, to the current flowing through the wire, its temperature was ascertained, the temperature resistance curve being previously known.

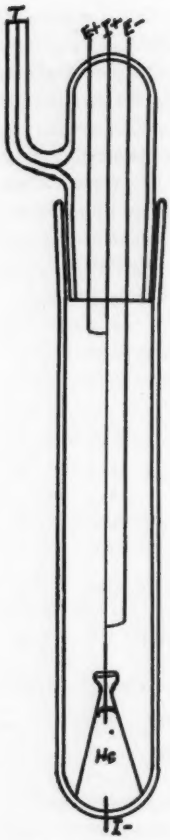


FIGURE 1

The resistance of a piece of the same wire was determined at various temperatures between 0° and 634°C. The wire was mounted on a mica support between very heavy copper leads and was contained in a tube of fused quartz. Constant temperatures were obtained by means of a fused salt bath heated by an electric resistance furnace. The temperatures were measured by means of a calibrated thermocouple, the potential across the wire with a potentiometer. The results checked the data of the Fansteel Products Co., who kindly supplied the wire.

The bulb was connected by means of a capillary tube, *T*, to a mercury manometer, to a mercury condensation pump, McLeod gauge, etc., and to reservoirs of ammonia, nitrogen, and hydrogen. The bulb was immersed in ice

The ammonia was synthetic ammonia from a cylinder, and was passed over solid potassium hydroxide. The hydrogen was electrolytic hydrogen from a cylinder and was passed over a heated platinum spiral, then over phosphorus pentoxide. The nitrogen, from a cylinder, was purified simply by passage over phosphorus pentoxide.

The course of the reaction was followed by measuring the accompanying change in pressure, the current being switched off and gases allowed to reach zero degrees before each reading. The theoretical endpoint was reached in a trial run.

Influence of the Pressure of Ammonia.—The reaction was more nearly of zero order than monomolecular, as seen from the following table showing the effect of the initial pressure of ammonia upon the time of half life.

EXPT. NO.	T = 1228° ABS. P (NH_3, MM.)	t (HALF LIFE, SECONDS)
9	50	65
21	50	67
53	100	120
8	100	114
52	200	210
24	200	203
51	300	272
45	300	286

A similar set of values was obtained when the time taken to reach 20% decomposition was compared with the initial pressure of ammonia. The

retarding effect of the nitrogen film (discussed later) must have been small in this case. The apparent order of the reaction is, therefore, almost zero. The true order of the reaction, i.e., the number of molecules actually participating in the decomposition, is probably one, by analogy to the decomposition on the other catalysts.

Influence of Products Previously Added.—Hydrogen was almost without influence upon the reaction at 1228° abs., when added in quantities up to 200 mm. of mercury. It produced, if anything, a slight accelerating effect.

Added nitrogen produced a marked retardation of the reaction velocity at 1228° abs. However, a preliminary investigation of the variation in the magnitude of the effect with increasing pressures of nitrogen did not reveal an orderly relationship. It was then found that the film of nitrogen upon the surface could not be pumped off in a vacuum, and that it persisted in a new experiment with no added nitrogen. In the further investigation of the effect of added nitrogen only those experiments were considered valid which were made after a standard experiment with no added nitrogen, and the wire at normal activity. The standard experiment was made at 1228° abs., with 100 mm. of ammonia. After being poisoned with nitrogen, the wire was found to return to its normal activity by performing one or more experiments with 100 mm. of ammonia, half of this amount being decomposed each time.

Proceeding in this manner, the remarkably smooth relationship shown in curve 1 was found to exist between the pressure of added nitrogen and the time of half life. The experiments were performed at 1228° abs. with 100 mm. of ammonia in each case. The points were obtained by averaging two or more experiments as follows.

EXPT. NO.	P (N_2)	HALF LIFE (SEC.)	AVERAGE
65	0	123	
72	0	126	
81	0	129	126
66	50	194	
68	50	188	191
70	100	250	
82	100	264	257
73	200	340	
86	200	346	343
76	300	350	
86	300	362	356

Traces of oxygen in the nitrogen could not have caused the retardation since molybdenum oxide is reducible at this temperature by hydrogen and hydrogen is formed in the decomposition. Retardation of the same magnitude was observed in two later experiments with nitrogen which

had been passed over metallic copper at red heat. Traces of moisture would surely distill from the wire at 950°C. to the walls at 0°C.

Temperature Coefficient.—Experiments were then performed at various temperatures between 1097° and 1228° abs., 100 mm. of ammonia being used in each case. Standard experiments were performed at intervals in order to check the catalytic activity of the wire. The results are shown in the following table.

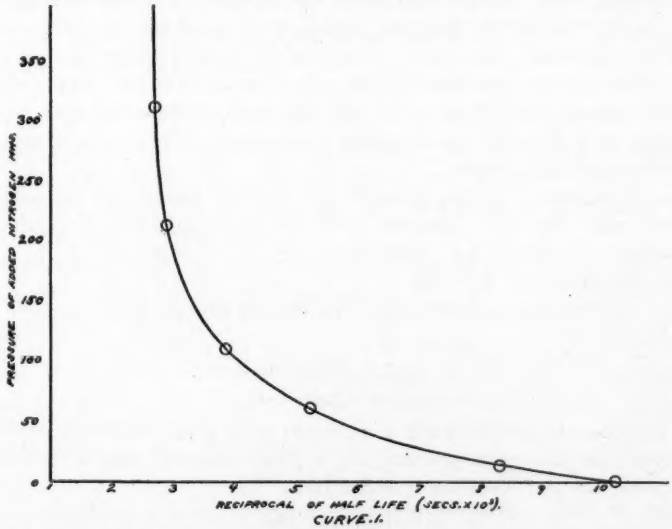
EXPT. NO.	t (HALF LIFE)	$T°$ ABS.	$1/T$	LN t (AVERAGE)
37	2015 sec.	1097	0.000912	7.5769
41	1892			
	1953.5 av.			
35	1080	1120	0.000893	6.89608
42	950			
54	935			
	988 av.			
34	680	1143	0.000875	6.4330
55	565			
	622 av.			
40	250	1184	0.0008448	5.6166
57	300			
	275 av.			
36	120	1228	0.0008147	4.8039
39	120			
43	132			
56	120			
	122 av.			

The ln T, $1/T$ points, when plotted, lie on a very good straight line. The slope of this line corresponds to a heat of activation of about 53,200 calories. Practically the same value of the heat of activation was obtained when the time required for fifteen per cent decomposition was taken as a measure of the rate instead of the time of half change. Perhaps the reason for this is that the fraction of the surface occupied by nitrogen is small in each case, as seen from curve 1. (The mean pressure of nitrogen in the case of the half-life measurements was 12 mm., and that for the fifteen per cent measurements 4 mm.)

The heat of activation as thus measured corresponds to the true heat of activation, since the reaction is of zero order even at the highest temperature and the retarding effect of the products can be neglected under the conditions of the experiments.

From the curve for the temperature coefficient it was calculated that 20% would be decomposed in 161 seconds at 1150° abs., when 100 mm.

of ammonia were used. In the decomposition on tungsten, 20% was decomposed in 161 seconds at 1129° abs. when 100 mm. of ammonia were used. The corresponding heats of activation are 53,000 for molybdenum and 39,000 for tungsten, which are, approximately, the true heats of activation. If the amounts of gas adsorbed were the same, in each case Q/T should be equal to Q_1/T_1, where Q and Q_1 are the heats of activation for the two metals, and T and T_1 the temperatures at which the reaction proceeds at the same rate on each.[3] While this is not exactly true, the result is not disappointing since the surface conditions could easily take care of the discrepancy. (Unfortunately, the dimensions of the tungsten wire were not available to the writer for comparison with those of the



molybdenum wire.) Perhaps something should be subtracted from the value for the heat of activation on molybdenum on account of the nitrogen film, even though it was not extensive.

Conclusions and Comparison of the Results with Those on Other Catalysts.—The influence of nitrogen in retarding the reaction is interesting and somewhat different from the retarding effects of products with platinum and silica as catalysts.

It was seen that the film of nitrogen persisted for some time even at 1228° abs. in the presence of pure ammonia. The union of the nitrogen with the metallic surface must, therefore, be strong. Since every molecule striking the surface of the wire condenses, as pointed out by Langmuir, it follows that if such a strong union were formed with every nitrogen

molecule, the surface would be almost completely covered with nitrogen, when its partial pressure is very small. That this is not the case is shown by curve 1. Therefore, only activated molecules of nitrogen are condensed in such a way as to form the poisoning film. The film may consist of atomic nitrogen.

The shape of curve 1 is that of the well-known adsorption isotherm with a saturation limit. However, a curve of this form would be obtained if the rate of condensation were proportional to the pressure of the gas. This, of course, would still be true if only the activated molecules were considered. (Values of $(1-s)$ are plotted in curve 1, whereas in the adsorption isotherm for nitrogen values of s would be plotted.)

It is seen from curve 1 that when the pressure of added nitrogen is great so that the surface is almost completely covered the rate of reaction does not approach zero, but a constant and considerable value. This may mean that the reaction still takes place on parts of the surface which are not poisoned by nitrogen, or that the reaction can still take place, although at a reduced rate, on a film of nitrogen. The kinetic treatment of these cases is as follows.

If s = fraction of surface covered with N_2 s_1 = fraction of the surface covered with NH_3, p = pressure of N_2 and p_1 = pressure of ammonia: then $k_1p(1-s-s_1) = k_2s$, and $k_1^1p_1(1-s-s_1) = k_2^1s_1$ (ref. 3, p. 152) from which $k_2s/k_1p = k_2^1s_1/k_1^1p_1$.

Since the reaction is of zero order with respect to ammonia.

$$s_1 = 1-s$$

$$s_1 = k_2sk_1^1p_1/k_2^1k_1p = 1-s$$

or

$$s = k_2^1k_1p/k_2^1k_1p+k_2k_1^1p_1.$$

p_1 which may be the pressure of ammonia at a given stage in any experiment, or the mean pressure for a given interval, was a constant. Therefore, the last equation is of the form

$$s = k_1p/k_2 + k_1p. \qquad (1)$$

This is the form the equation for the fraction of the surface covered by nitrogen would assume if Langmuir's theory of adsorption held, and if no ammonia were present.

Now the rate of reaction must be proportional to $s_1 = 1-s$. Therefore, the rate of reaction must be proportional to $k_2/k_2 + k_1p$, which is the value of $(1-s)$ from equation (1). On the assumption that there are portions of the surface not poisoned by nitrogen and upon which ammonia can decompose, and that the Langmuir theory of adsorption held, the rate of reaction would be expressed by the equation

$$\text{Rate of reaction} = (k_3k_2/k_2 + k_1p) + k_4. \qquad (2)$$

In the experiments on the effect of added nitrogen, the pressure of ammonia was kept constant and, therefore, it is permissible to use the

time of half life as a measure of the rate of reaction. On evaluating the constants from the first four points of curve 1, equation (2) becomes

$$\text{Reciprocal half life} = (0.00865/0.0214p + 1) + 0.00155. \quad (3)$$

This equation fits the results given in curve 1 very well, as the following table shows.

RECIPROCAL HALF LIFE FOUND	RECIPROCAL HALF LIFE CALCULATED	MEAN PRESSURE OF NITROGEN
0.0102	0.0101	0
0.0083	0.0084	12
0.0052	0.0052	62
0.0039	0.0041	112
0.0029	0.0031	212
0.0027	0.0026	312

On the assumption that reaction takes place on top of the film of nitrogen, the kinetic equation would be

$$\text{Reciprocal half life} = (k_3k_2/k_1p + k_2) + (k_4k_1p/k_2 + k_1p). \quad (4)$$

This equation, by rearrangement, assumes the same form as equation (3), and contains the same number of constants. It is, therefore, impossible to distinguish between the two mechanisms by testing their theoretical kinetic equations.

The following table gives a comparative summary of the decomposition of ammonia upon the various catalysts.

CATALYST	APPARENT ORDER OF REACTION	EFFECT OF N_2	EFFECT OF H_2	HEAT OF ACTIVATION (CALORIES)
Silica	1	Rate prop. to (1-c[products])		86,000
Platinum	1	None	Rate prop. to $1/[H_2]$	140,000
Tungsten	0	None	None	39,000
Molybdenum	0	Rate prop. to $(k/pN_2 + k_1) + k_2$	None	53,200

The comparison is consistent in all respects in the light of present theories of heterogeneous kinetics.[3]

These experiments display two points of interest to the general study of catalytic action. Firstly, they give, in conjunction with the previous experiments on tungsten, support to the proposition that the magnitude of the heat of activation determines the rate of a catalytic reaction, as well as an uncatalyzed reaction, when secondary effects are not considered. Secondly, they furnish one more example of the importance of retarding films in the practical operation of catalytic reactions.

Summary.—The thermal decomposition of ammonia was investigated upon a heated molybdenum wire and compared with the results on other catalysts.

The decomposition was apparently of zero order.

The reaction was unretarded by hydrogen but was retarded by nitrogen.

At a given temperature, and pressure of ammonia, the rate of reaction could be expressed by the equation, reciprocal of half life = $(k_3k_2/k_2 + k_1p\mathrm{N}_2) + k_4$. This equation was derived theoretically.

The observed heat of activation for the decomposition was found to be 53,200 calories, which was shown to be the true heat of activation.

The writer wishes to express his best thanks to Prof. W. D. Bancroft for his helpful interest and suggestions, to the National Research Council for a Research Fellowship and to Cornell University for apparatus and laboratory facilities.

[1] NATIONAL RESEARCH FELLOW IN CHEMISTRY.

[2] Hinshelwood and Burk, *J. C. S.*, **127**, 1105 (1925). References to other work are contained in this paper.

[3] Hinshelwood, *The Kinetics of Chemical Change in Gaseous Systems*, Oxford University Press (1926), p. 92. The same argument would apply to monomolecular reactions on surfaces since the fraction of adsorbed molecules receiving the heat of activation Q would also be given, approximately, by $e^{-q/RT}$.

ESKIMOS AND PLANTS

BY A. V. KIDDER

NATIONAL RESEARCH COUNCIL

Communicated January 8, 1927

In a recent paper entitled "Persistence of Plants in Unglaciated Areas of Boreal America,"[1] Professor M. L. Fernald, of Harvard, has sought to demonstrate that certain generally accepted theories of the distribution of arctic floras are not tenable. It has been believed, for example, that during the Glacial Period the cold of boreal America was so intense and uninterrupted that plant life left the north and gradually migrated south. Upon the retreat of the glacial fields it returned north. Fernald adduces geological evidence to show that glaciation of continental ice-sheet type did not prevail over the Arctic Archipelago, the Torngat Mountains of Northeastern Labrador, the Long Range of Western Newfoundland nor the Shickshock Mountains of Gaspe Peninsula. He further points out that the above regions are now occupied by certain plant species of extreme specialization and, therefore, presumably great local antiquity; species, moreover, which are not found in neighboring areas once glaciated. He holds that because these species have not migrated to occupy now equally favorable but formerly glaciated districts in the immediate vicinity of their present habitats, it is unreasonable to suppose them capable of the vastly longer migrations called for under the old theory of retreat before the glaciers and return in post-glacial times. He considers, there-

fore, that these highly specialized plants are remnants of a general preglacial flora which survived in the far north in districts not covered during the Pleistocene by continental ice.

Fernald's paper indicates the strong probability of the existence, during the Glacial Period, of areas in boreal America by no means untenable by man (parts of Greenland now inhabited are presumably less favorable for human occupancy than were, during the Pleistocene, the areas under consideration). The Eskimos, with their great cultural and particularly their extreme physical specialization, would appear to be comparable in many ways to Fernald's relic plants. Furthermore, there is some evidence that the Eskimos have spread both east and west from a region more or less central to their present range, a region not far distant from that discussed by Fernald. The foregoing is not intended as the postulation of a theory to account for the peculiarities of Eskimo distribution, culture or physique; it is desired merely to call attention to a research probably not generally known to anthropologists. Were reliable evidence of preglacial man ever to be found in the New World, the districts pointed out by Fernald would become of great interest. In any case it might not be amiss to keep an eye on them archaeologically.

[1] *Mem. Amer. Acad. Arts Sci., Boston,* **15,** 1925 (237–342).

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*CHROMOSOME AND GENE MUTATIONS IN DATURA FOLLOWING EXPOSURE TO RADIUM RAYS**

BY C. STUART GAGER AND A. F. BLAKESLEE

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Communicated January 14, 1927

At the Boston, 1922, meeting of the Botanical Society of America, the writers presented the results of a preliminary study of the effects upon the offspring of exposing flowers of Datura to radium rays. At that time there was not available a sufficient body of data in regard to the mutability of untreated parents to permit us properly to evaluate the significance of the results. We have now, however, a considerable accumulation of records regarding the occurrence of both gene and chromosomal mutations in closely comparable normal material which can be handled as control to the treated material. Moreover, the analysis of the chromosomal constitution of Nubbin lends a peculiar interest to this mutant which arose from a radium-treated parent. Since it appears not unlikely that the radium treatment was responsible for the breaking up and the reattachment

of parts of non-homologous chromosomes in the formation of Nubbin, which is described in the succeeding paper, it seems desirable at this time to publish the data we have so far obtained in the form of a preliminary contribution.

It is hoped, however, that it may be possible, soon, to take up again the study of the effects of radium treatment upon the genetic constitution of the offspring and to determine more precisely at what stage or stages the stimulus is effective.

The present paper is an outgrowth of experiments of the senior author carried out in 1906 in which he exposed pollen grains and egg cells of *Oenothera biennis* to radium rays. The effects obtained seemed confined to purely somatic characters of the offspring and did not appear to affect their genetic constitution. Somewhat similar results were later obtained by Emmy Stein working with Antirrhinum.[2]

The radium preparations used in the present study consisted of small sealed glass tubes, of capillary bore, containing radium emanation whose activity from day to day during its period of "decay" is known by the application of a formula. For these preparations the authors are indebted to the Memorial Hospital, New York City, and the personal cooperation of Dr. Halsey J. Bagg of the hospital staff.

It may be stated that radium emanation is a radio-active gas, giving off α and β particles and γ-rays. The latter are a penetrating type of X-ray and, since the α and β particles are largely or wholly absorbed by the walls of the sealed glass tubes employed, the physiological effects observed in our experiments are to be attributed to the action of γ-rays. In the experiments to be described, flower buds of plants of *Datura Stramonium* belonging to line 1, which had been inbred by selfing for nine generations, were exposed to γ-rays by inserting in them sealed glass tubes containing radium emanation. Exposures were made with young buds of varying ages and with radium preparations of varying strength in terms of microcuries (mc.) for varying lengths of time. In some cases the tube was inserted into the center of the ovary cell, in other cases into the wall or partition between the cells and in still other cases it was merely placed in contact with the outer wall of the ovary. Several exposures of anthers were made at about the time of reduction in pollen mother-cells but in each case the bud dropped off before anthesis.

The experiments from which offspring were obtained will be referred to by their original numbers although not repeated here in numerical sequence.

Experiment 5. May 3, 1921. Plant No. 202(40). Strength, 13 mc.; time of exposure 15 min., ovary about 5 mm. long and corolla fallen away showing that fertilization had taken place. The tube was inserted in the partition between cells 1 and 2. By June 22nd the ovary had become

about 15 mm. long, had ripened and was dehiscing in normal manner. Cell 1 contained only one seed; cell 2, two seeds; cell 3, eleven seeds; cell 4, eight seeds and one small, seed-like body (aborted ovule). When these seeds were planted, the single seed from cell 1 failed to germinate; the two seeds from cell 2 both germinated, nine of the eleven seeds from cell 3 and three of the eight seeds from cell 4 germinated. Of the total 14 plants, all were normal in appearance.

Experiment 2. April 23, 1921. Plant 202(40). Strength 13 mc.; time of exposure 10 min. Ovary about 10 mm. in diameter; corolla fallen away. The tube was inserted into the midst of cell number 1. On June 9th the ovary was found ripe and dehisced and seeds were collected, and gave germinations as follows: Cell 1 had no seeds, cell 2 had 21 seeds of which 7 germinated, cell 3 had 25 seeds of which 15 germinated, cell 4 had 13 seeds of which 9 germinated. None of the 31 plants obtained in experiment 2 departed from the normal in appearance.

Experiment 1. April 22, 1921. Plant 202(15). Strength 13 mc.; time of exposure, 10 min.; corolla tube about $1^1/_4$ in. long with limb showing blue color. At time of treatment, reduction had certainly already taken place in the pollen-mother-cells and almost certainly also in the megaspore-mother-cells. On June 22nd, the ovary was mature. Cell number 1, into the midst of which the tube was inserted, was partially atrophied and bore only eight seeds. Seeds were collected and germinated as shown in the adjoined table.

CELL NO.	NUMBER OF SEEDS	TOTAL SEEDLINGS	NORMALS	CHROMOSOMAL MUTANTS	PER CENTS CHR. MUTS.
1	8	7	5	2	28.57
2	33	24	18	6	25.00
3	68	56	47	9	16.07
4	40	26	23	3	11.54
Totals	149	113	93	20	17.70

Following are the chromosomal types discovered in the immediate progeny of the treated parent. They are mostly $2n + 1$ forms. Figures in parentheses indicate the number of F_2 individuals from them grown to recordable age. Cell 1 gave rise to one $2n + 2$ Globe (0) and to one Nubbin (24) [cf. next paper]. Cell 2 gave rise to 1 Buckling (112), 1 Poinsettia (123), 1 Cocklebur (89), 1 Spinach (0), 1 Microcarpic (196), 1 chromosomal type (44). This latter plant was recorded as an off-type and had a high proportion of bad pollen grains. Its seeds showed poor germination but gave 40 normals, 2 Rolleds and 2 Bucklings. It was undoubtedly an extra-chromosomal type and probably Rolled or Rolled-Buckling. Rolled often comes through in low proportions. Cell 3 gave rise to 1 Globe (130), 1 Globe (13), 1 Cocklebur (0), 1 Rolled (118), 1 Rolled (83), 1 Buckling (68), 1 Strawberry (15)—a secondary of Buckling—

1 Spinach (20), 1 Spinach (7). Cell 4 gave rise to 1 Poinsettia (109), 1 Ilex (85), 1 double mutant Buckling-Microcarpic (96).

A percentage of 17.7 chromosomal mutants in over 100 offspring from a single capsule is enormously greater than we have ever obtained before or since. There was no tendency in plant 202(15) toward chromosomal mutations, since an untreated capsule gave 36 offspring—all of them normal. A total of 73 ($2n + 1$) forms were obtained in a total of 15,417 progeny from untreated normal diploid parents in line 1*A* or an average of 0.47 per cent. In view of the above figures, we believe the radium treatment was responsible for the increased proportion of chromosomal mutations, as also for the appearance of the compound chromosomal type Nubbin.

Two gene mutations occurred in the formation of offspring from the treated plant 202(15). The Microcarpic plant from cell 2 when selfed threw 162 normals and 34 albinos, showing that this Microcarpic parent was heterozygous for the albino gene. This character, albino, is not a convenient one with which to work, since it is not viable beyond the seedling stage when in the homozygous condition and the segregation ratios have been somewhat irregular, due doubtless to poor germination of the albino seedlings. It is to be classed, however, as a simple gene mutant.

The Ilex individual from cell 4 produced when selfed—62 normals, 20 Ilex individuals, 1 Globe and 2 diploids of the same peculiar appearance which are representatives of the new mutant—"Swollen," a character with thick, *swollen*, brittle stems and petioles, thick leaves, and dwarf habit of growth. The Ilex parent was heterozygous for the gene swollen and the reason the character did not segregate out in a 1:3 ratio is due to the fact determined later that the gene for Swollen is located in the Ilex chromosome.

Unfortunately, in the experiment, we were looking only for chromosomal mutations and did not save seeds of any of the normal offspring from the treated capsule to see if they were heterozygous for other new genes. The number of offspring obtained from the chromosomal forms are indicated in parentheses in a preceding paragraph. In many cases they are too few to show a new character by segregation. We may take them as they stand, however, and show the results in the adjoined table.

	NUMBER OF OFFSPRING TESTED	NUMBER OF OFFSPRING HETEROZYGOUS FOR NEW GENE	PERCENTAGE OF GENE MUTATIONS
Cell 1	1	0	0
Cell 2	5	1 (albino)	20.00
Cell 3	8	0	0
Cell 4	3	1 (swollen)	33.33
Totals	18	2	11.11

Two new genes in only 18 parents is a small number absolutely, but is very large if the proportion of gene mutations can be considered significant

with so few individuals tested. Aside from these two mutations from this treated capsule and three or four mutations which occurred in F_1's from a haploid, we have not yet determined any gene mutations in *Datura Stramonium*. There were tested by selfing about 400 individual parents in the F_2 or later generations from the haploid 1*A* without discovering a single gene mutation.

Following treatment with radium emanation, we have obtained in the offspring from a single capsule—(*a*) 17.7 per cent chromosomal mutants (chiefly non-disjunctional forms[3]), a much higher percentage than ever obtained from untreated capsules, the average for over 15,000 offspring being 0.47 per cent; (*b*) a new compound chromosomal type, Nubbin; (*c*) two new gene mutants out of 18 individuals tested.

It is our belief that for most, if not for all, of these three types of results, the radium treatment may be held largely responsible.

* Brooklyn Botanic Garden Contributions, No. 49.

[1] Gager, C. Stuart, "Effects of the Rays of Radium on Plants," *Mem. New York Bot. Gard.*, **4**, 1–278, 1908. Results also presented before Sect. G, *A. A. A. S.*, Jan., 1907.

[2] Stein, Emmy, "Ueber den Einfluss von Radiumbestrahlung auf Antirrhinum," *Zeitschr. Indukt. Abstamm. Vererb.*, **39**, 1–15, 1922. Also *Ibid.*, **43**, 1–87, 1926.

[3] Cf. Mavor, James W., "The Production of Non-Disjunction by X-Rays," *Science*, **55**, 295–297, 1922.

THE CHROMOSOMAL CONSTITUTION OF NUBBIN, A COMPOUND (2n + 1) TYPE IN DATURA[1]

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Nubbin is one of the mutant types of the Jimson Weed (*Datura Stramonium*) which has been shown[2] to have a single extra chromosome. It was found in 1921 among the offspring of a plant which had been treated

TABLE I

OFFSPRING OF NUBBIN, PINCHED AND HEDGE FROM SELFINGS OR FROM BACK-CROSSES LINE 1. GARDEN RECORDS. FIGURES IN PARENTHESES REPRESENT PERCENTAGES UNRELATED TYPES IN OFFSPRING NOT SHOWN

TYPES	NO. OF PARENTS	TOTAL	?	2n	Nb	Pn	Bk	Hg	Ec	Rl
Nubbin	7	1226	5	668	424	58	9	36	14	...
1923–26				(54.5)	(34.6)	(4.7)	(0.7)	(2.9)	(1.1)	...
Pinched	4	561	13	339	...	180	10	...	1	8
1924–26				(60.4)	...	(32.0)	(1.8)	...	..	(1.4)
Hedge	3	689	6	541	...	...	..	121	11	5
1924–26				(78.5)	...	...	..	(17.6)	(1.6)	(0.7)

by Dr. C. Stuart Gager with radium emanation (cf. preceding paper) and has not appeared spontaneously since. Its parent was a normal ($2n$) diploid in line 1 which had been inbred by selfing for nine generations.

Like other ($2n + 1$) forms, Nubbin does not breed true, being not at all transmitted through the pollen and transmitted through the egg cells to considerably less than the 50 per cent theoretically possible. It differs from the other ($2n + 1$) forms in the types which it throws in its offspring as shown in figure 1, in which arrows point to capsules of the types of offspring produced. The normal ($2n$) offspring are omitted from the diagram. Nubbin (Nb) regularly throws Nubbins but in addition two other new types with single extra chromosomes, Pinched (Ph) and Hedge (Hg) which have never appeared spontaneously in our cultures. Nubbin also throws a few of the primary ($2n + 1$) types, Buckling (Bk) and Echinus (Ec). Pinched throws Pinched, Buckling and the primary ($2n + 1$) Rolled (Rl) while Hedge throws Hedge, Echinus and Rolled. Table 1 gives the offspring of Nubbin, Pinched and Hedge within line 1.

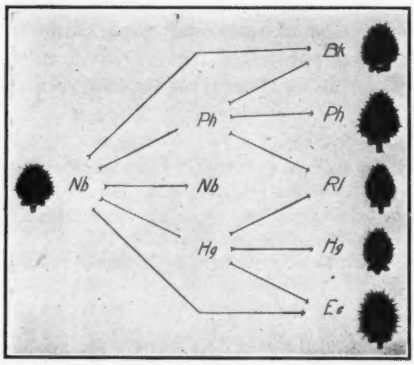

FIGURE 1
Diagram with capsules illustrating the breeding behavior of Nubbin, Pinched and Hedge.

From the breeding behavior we may conclude that Nubbin cannot be a primary ($2n + 1$) form (5, 6) in which the extra chromosome is similar in makeup to the other two members of the set since (*a*) primary types do not regularly throw other specific mutants and (*b*) primaries are regularly thrown by triploids while Nubbin is not. Furthermore, Nubbin cannot be a secondary of the ordinary type since secondaries throw their own primaries alone in any regular proportions.

Buckling bears the same relation to Pinched that Echinus does to Hedge. Buckling and Echinus are both thrown by Nubbin and hence bear the same relation to Nubbin.

Could the occurrence of Pinched and Hedge be explained in some other way, the fact that Nubbin throws both Buckling and Echinus might lead us to consider it a double mutant, Echinus-Buckling, with an extra chromosome in both the Echinus and in the Buckling set. Even if the chromosomal counts could allow Nubbin to be considered a ($2n + 1 + 1$) mutant, the proportion of types in its offspring is unlike that from a double

mutant. In the progeny of Echinus-Buckling, diploids, the two $(2n + 1)$ types and the double type average a rough approximation to a 6:2:2:1 ratio. Echinus-Buckling throws fewer double mutants than it does the simple mutants—Echinus and Buckling—while Nubbin throws more Nubbins than it does Buckling or Echinus types.

From figure 1 it will be seen that although Nubbin itself does not throw Rolled, it is related in some way to Rolled through the fact that both the reciprocal mutants, Pinched and Hedge, throw Rolled. Pinched and Hedge are also morphologically related reciprocally to the two secondaries of Rolled, as will be shown later. Nubbin is related to Rolled in still another way through its behavior when heterozygous for "*B*" whites. It has been pointed out that when all the primary $(2n + 1)$ types are rendered heterozygous for certain white-flowered lines, called "*A*" whites

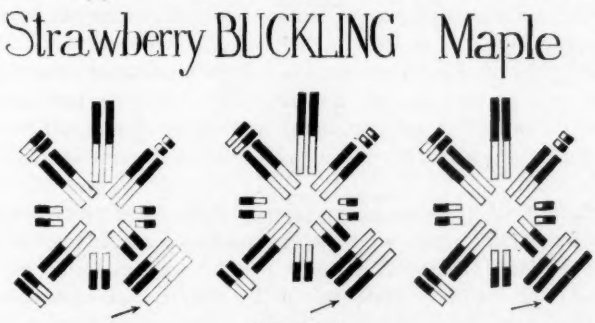

FIGURE 2

Chromosomal diagrams of types in the Buckling group. Buckling is the primary, Strawberry one of its secondaries and Maple probably the other secondary.

the mutant type Poinsettia alone gives trisomic ratios in its offspring, indicating[3] that the factors for purple and white are located in the Poinsettia chromosome. When, however, these same primaries are heterozygous for certain other white lines called *B* whites, trisomic ratios are thrown by Rolled as well as by Poinsettia. The responsibility for the ratio abnormalities has been traced to the Rolled chromosome[4] and a hypothesis in explanation has been offered by the cytological investigations of Dr. Belling.[2] It will be sufficient to state that Nubbin when heterozygous for *A* whites gives normal disomic ratios while, when it is heterozygous for *B* whites, it throws a considerable excess of whites in its offspring.

Before discussing the morphology of Nubbin, it will be necessary to point out some of the characteristics of the related types. The distinction between primaries and secondaries has been discussed in earlier publications.[5,6]

The Buckling Group consists of the primary Buckling and of the secondaries Strawberry and Maple. The extra chromosome in Buckling is like the other two members of the Buckling set. In the secondaries, however, the extra chromosome is a double half chromosome. Speaking in terms of the conventionalized diagrams of figure 2, the extra chromosome of Strawberry is made up of two "white" halves while the extra chromosome of its complementary secondary is made up of two "shaded" halves. It is for this reason that in this as in other groups the primary is intermediate in characters between its two complementary secondaries, having extra both a white and a shaded half. The secondaries, on the other hand, are extreme in certain characters having the factors in one half chromosome doubled and the factors in the other half as in normal $2n$ plants. By taking stock of the peculiarities of the three types in this group we can learn what are the aggregate of factors not only in the whole Buckling chromosome but also in its two halves. Thus we conclude that the Strawberry half of Buckling contains factors tending to make the plant erect, and the leaves relatively narrow, while the Maple half of Buckling contains factors for spreading habit and broad leaves. The primary Buckling is intermediate in respect to all these characters as well as others not mentioned.

The Rolled Group consists of the primary Rolled and its two secondaries Sugarloaf and Polycarpic, while the *Echinus Group* consists of the primary Echinus and its secondary Mutilated.

Nubbin has distinctly dimorphic pollen due to lack of starch in half the grains as have also only Echinus and its secondary Mutilated. It must, therefore, possess in excess either an Echinus whole chromosome or its Mutilated half, in which the factor or factors for the dimorphism are located. If an extra whole Echinus chromosome is present, there must also be an extra whole Buckling chromosome since these two primaries have the same relation to Nubbin. This would make Nubbin morphologically the double mutant, Echinus-Buckling. The morphology as well as the breeding behavior shows that Nubbin cannot be Echinus-Buckling.

Since a whole extra Echinus chromosome is ruled out, Nubbin must contain the Mutilated half chromosome in excess since both Mutilated and Nubbin have dimorphic pollen. Nubbin is much more vigorous in growth than the mutant Mutilated which has as its extra chromosome two similar halves containing the factors for dimorphism. Hence, Nubbin can contain, at most, a single Mutilated half chromosome in excess, otherwise the plant would be more severely unbalanced. Possessing an extra half chromosome of Echinus, Nubbin should be expected to throw a certain proportion of this primary as does also the secondary Mutilated. We have apparently discovered the chromosomal component responsible

for the dimorphic pollen in Nubbin and for the relation which Nubbin bears to the primary Echinus. By a similar line of argument, taking into account the fact that the capsule of Nubbin most nearly resembles that of Strawberry, it can be shown that a single Strawberry half of the Buckling chromosome is in excess in Nubbin.

Morphologically, therefore, Nubbin seems to be $2n$ + $^1/_2$Mutilated + $^1/_2$Strawberry. Its pollen dimorphism and thickish leaves come from the Mutilated component while its capsule shape and erect habit come largely from the Strawberry component.

In considering Nubbin to be morphologically $2n$ + $^1/_2$Mutilated + $^1/_2$ Strawberry, as we are forced to do, we have not explained the relation of Nubbin to Rolled nor to Pinched and Hedge. The Rolled chromosome has been determined by Dr. Belling to be distinctly the largest of the twelve[7] and either half as well as the whole chromosome has a very distinct effect upon the appearance of a plant when present in excess. It seems clear that morphologically Nubbin does not contain an extra Rolled chromosome nor an extra half of this chromosome. A way out of the difficulty may be found by a consideration of the morphology of the reciprocal types—Pinched and Hedge.

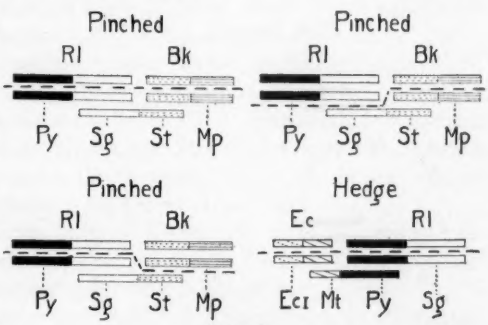

FIGURE 3

(*a*) above, left. Diagram of chromosomes in Pinched. The dotted line represents the disjunction at reduction which leads to the formation of ln and Pinched eggs and pollen grains.

(*b*) lower, left. Dotted line represents disjunction leading to formation of (n + 1) Buckling and ln − lBk + $^1/_2$St $^1/_2$Sg eggs and pollen grains. The latter cells, since they lack a half of the Buckling chromosome, would probably die and be represented by aborted ovules and pollen grains.

(*c*) above, right. Dotted line represents the disjunction leading to formation of (ln + 1) Rolled and ln − 1 Rl + $^1/_2$St $^1/_2$Sg eggs and pollen grains. The latter cells die.

(*d*) lower, right. Diagram of Hedge. Ec II is the half of the Echinus chromosome opposite to Mutilated. The dotted line represents the disjunction leading to formation of ln and Hedge eggs and pollen grains. Rolled and Echinus eggs and pollen grains would be formed by a type of disjunction similar to that shown in figure 3, (*b*) and (*c*), for the formation of Rolled and Buckling gametes in the mutant Pinched.

Pinched strongly resembles Sugarloaf, the secondary of Rolled, and less markedly Strawberry, the secondary of Buckling, and is judged to have in its extra chromosome the Strawberry half of the Buckling chromosome

joined to the Sugarloaf half of the Rolled chromosome, justifying the chromosomal formula $2n + {}^1/_2\text{St}\ {}^1/_2\text{Sg}$. Chromosomal diagrams are given for Pinched (Fig. 3) showing the probable method of disjunction leading to the formation of Buckling and Rolled egg cells and pollen grains.

Similarly Hedge shows resemblances to Polycarpic, the secondary of Rolled (complementary to Sugarloaf) and also to Mutilated, the secondary of Echinus. It is believed to have in its extra chromosome, the Mutilated half of the Echinus chromosome joined to the Polycarpic half of the Rolled chromosome justifying the formula $2n + {}^1/_2\text{Mt}\ {}^1/_2\text{Py}$ (Fig. 3).

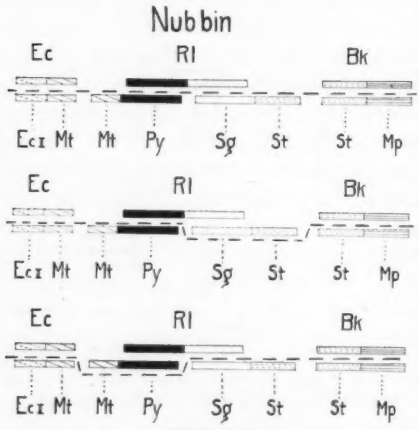

FIGURE 4

(*a*) Above. Diagram of chromosomes in Nubbin. The horizontal dotted line represents the disjunction which leads to the formation of In and Nubbin (In − lRl + ${}^1/_2\text{Sg}\ {}^1/_2\text{St}$ + ${}^1/_2\text{Mt}\ {}^1/_2\text{Py}$) eggs and pollen grains.

(*b*) Middle. The dotted line represents the disjunction which leads to the formation of Pinched and In − lRl + ${}^1/_2\text{Mt}\ {}^1/_2\text{Py}$ eggs and pollen grains. The latter cells, since they lack the Sugarloaf half of the Rolled chromosome, would abort.

(*c*) Below. The dotted line represents the disjunction which leads to the formation of Hedge and In − lRl + ${}^1/_2\text{St}\ {}^1/_2\text{Sg}$ eggs and pollen grains which latter cells would be expected to abort. Buckling and Echinus, but not Rolled, gametes could be formed by a type of disjunction indicated under Pinched and Hedge.

Nubbin may be considered to be compounded of Pinched and Hedge, having as extras both the Pinched and the Hedge compound chromosome (which together furnish the two opposite halves of the Rolled chromosome), and lacking a single whole Rolled chromosome. Morphologically, therefore, Nubbin should show no Rolled characters since it would possess the factors of only two Rolled chromosomes. Its chromosomal formula may be written $2n - 1\text{Rl} + {}^1/_2\text{St}$-${}^1/_2\text{Sg} + {}^1/_2\text{Mt}{}^1/_2\text{Py}$. Chromosomal diagrams (Fig. 4) are given for Nubbin, showing its probable constitution and the methods of disjunction that might lead to the formation of Normal, Nubbin, Pinched and Hedge gametes.

For each Pinched and Hedge pollen grain formed by Nubbin, there could be expected an aborted grain. The fact that Nubbin shows about 13 per cent aborted pollen grains while Pinched and Hedge show only about 7 per cent[8] is in agreement with the arrangement of chromosomes

shown in the diagrams and with the breeding behavior shown in table 1. The diagrams are also in accord with the attachment of non-homologous chromosomes shown by Dr. Belling[2] for Nubbin and Pinched and Hedge.

The only fact so far discovered which seems to be in conflict with our interpretation of the chromosomal constitution of Nubbin, Pinched and Hedge is the lack of strong pollen dimorphism in Hedge which is believed to contain in excess the Mutilated half of the Echinus chromosome. The majority of the chromosomes must favor starch production since in diploids and in all primary $(2n + 1)$ types, except Echinus, all the pollen grains contain an abundance of starch. It is not unreasonable to assume, therefore, that the Polycarpic half of the Rolled chromosome may contain starch-favoring factors antagonistic to those in the Mutilated half of Echinus which are responsible for the reduction of starch.

The present paper presents in brief a method of analysis as well as contributes to the solution of a special problem in Datura. A fuller treatment of the subject will be given in another publication.

Reference has been made to investigations in progress as to the nature of "*A*" and "*B*" white lines. In addition certain lines when crossed together produce F_1's with a definite proportion of aborted ovules and pollen grains.[9] It is believed that responsibility for the peculiarities of these races may be laid to chromosomal changes similar to those which appear to have occurred in the production of Nubbin.

[1] Abstract of paper presented before the International Congress of Plant Sciences, Aug. 18, 1926, awarded the A. Cressy Morrison Prize No. II of the New York Academy of Sciences, December 20, 1926.

[2] Belling, J., and A. F. Blakeslee, 1926, these PROCEEDINGS, **12**, 7–11.

[3] Blakeslee, A. F., and M. E. Farnham, 1923, *Amer. Nat.*, **57**, 481–495.

[4] Blakeslee, A. F., 1924, quoted in *Yearbook of Carnegie Institution of Washington*, No. **23**, pp. 24–27.

[5] These PROCEEDINGS, 1924, **10**, 109–116.

[6] Belling, J., and Blakeslee, A. F., 1924, these PROCEEDINGS, **10**, 116–120.

[7] Belling, J., quoted in *Yearbook of Carnegie Institution of Washington*, No. **23**, 1924, pp. 28–30.

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